

MODULI OF LOG TWISTED $\mathcal{N} = 1$ SUSY CURVES

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ABSTRACT. The goal of the present paper is to construct a smooth compactification of the moduli superstack classifying pointed $\mathcal{N} = 1$ SUSY (= SUSY₁) curves. This construction is based on the Abramovich-Jarvis-Chiodo compactification of the moduli stack classifying spin curves. First, we give a general framework of a theory of log superschemes (or more generally, log superstacks). Then, we introduce the notion of a pointed (stable) log twisted SUSY₁ curve; it may be thought of as a logarithmic and twisted generalization of the classical notion of a pointed SUSY₁ curve, as well as a supersymmetric analogue of the notion of a pointed (log) twisted curve. The main result of the present paper asserts that the moduli superstack classifying pointed stable log twisted SUSY₁ curves may be represented by a log superstack whose underlying superstack is a superproper and supersmooth Deligne-Mumford superstack. Consequently, this moduli superstack forms a smooth compactification different from the compactification proposed by P. Deligne.

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INTRODUCTION

0.1. The goal of the present paper is *to provide a rigorous construction of a smooth compactification of the moduli superstack classifying pointed $\mathcal{N} = 1$*

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SUSY curves. Throughout the present paper, we abbreviate “ $\mathcal{N} = 1$ SUSY” to “SUSY₁” for simplicity. Recall that SUSY₁ curves and their analytic counterparts, called super Riemann surfaces, have been widely considered (intensively in the 1980s) in the physical literature on supersymmetry. Super Riemann surfaces are defined to be complex supermanifolds of superdimension $1|1$ satisfying an additional superconformal condition (cf. e.g., [15]). Super Riemann surfaces play a role of the correct supersymmetric analogue of Riemann surfaces, and their moduli superspace plays a role analogous to the role of moduli space classifying Riemann surfaces in bosonic string theory (cf. [8]). Indeed, just as the world sheet of a bosonic string carries the structure of a Riemann surface, the world sheet in superstring theory is a super Riemann surface. Also, perturbative calculations in superstring theory are carried out by integration over this moduli superspace. Besides having such physical applications, the theory of super Riemann surfaces and their moduli is interesting on its own from the mathematical viewpoint. In order to achieve a deep understanding of this theory (from the mathematical viewpoint or another), it will be worth asking the following question regarding their global structure that should be answered:

What is a natural (smooth) compactification of the moduli superspace classifying super Riemann surfaces (or more generally, pointed SUSY₁ curves)?

0.2. To obtain an answer of this question, P. Deligne constructed, in his letter to Y. Manin (cf. [6]), a smooth compactification of the moduli superstack classifying (unpointed) SUSY₁ curves; it may be thought of as an analogue of the Deligne-Mumford compactification of the moduli stack classifying proper smooth algebraic curves, and obtained by adding certain divisors at infinity parametrizing SUSY₁ curves with nodes. The main difference from the bosonic case is that, in constructing the compactification, we need to allow two different types of degeneration of SUSY₁ curves, called Neveu-Schwarz and Ramond degenerations. We refer to [20], §6, for a detailed exposition, including the physical viewpoint, of the moduli superspace classifying SUSY₁ curves (with marked points). In the present paper, we consider a smooth compactification different from the compactification constructed by P. Deligne, including the case of pointed SUSY₁ curves.

0.3. Let us describe the main theorem of the present paper. Let S_0 be a noetherian affine scheme over $\mathbb{Z}[\frac{1}{2}]$, λ a positive even integer which is invertible in S_0 , and (g, r) a pair of nonnegative integers such that r is even and $2g - 2 + r > 0$. Write $\mathfrak{Sch}_{S_0}^{\text{log}}$ (cf. (38)) for the category of fs log superschemes (cf.

Definition 2.1.2) over S_0 . Also, write

$$(1) \quad {}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$$

(cf. (119)) for the category fibered in groupoids over $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ classifying families of *stable log twisted SUSY₁ curves of type (g, r, λ)* (cf. Definition 3.6.1) parametrized by log superschemes in $\mathfrak{Sch}_{/S_0}^{\otimes \log}$. Then, our main result is the following theorem. (See Definitions 1.6.5 (ii), 1.7.1, 1.7.2, and 1.2.4 for the definitions of various notions appeared in the statement.)

Theorem A.

${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ may be represented by a log superstack whose underlying superstack is a superproper and supersmooth Deligne-Mumford superstack over S_0 of relative superdimension $3g - 3 + r|2g - 2 + \frac{r}{2}$.

0.4. Let us make a remark on the main result just described. Denote by ${}^{\S_1}\mathfrak{M}_{g,r}^{\otimes}$ the moduli superstack classifying r -pointed (supersmooth) SUSY₁ curves of genus g (in the classical sense) over log superschemes in $\mathfrak{Sch}_{/S_0}^{\otimes \log}$. Since any pointed (supersmooth) SUSY₁ curve is a stable log twisted SUSY₁ curve, we have a natural inclusion ${}^{\S_1}\mathfrak{M}_{g,r}^{\otimes} \hookrightarrow {}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$; it is an open immersion whose image is dense and coincides with the locus in which the log structure of ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ becomes trivial. Thus, the moduli log superstack ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$, being our central character, forms a smooth compactification of ${}^{\S_1}\mathfrak{M}_{g,r}^{\otimes}$ (different from the compactification by P. Deligne). The feature of our compactification is that we add divisors at infinity which parametrize pointed SUSY₁ curves (equipped with a logarithmic structure) admitting at most a single type of degeneration.

Next, recall the discussion of the non-projectedness (cf. [7], § 2), as well as the non-splitness (cf. Definition 1.4.1), of ${}^{\S_1}\mathfrak{M}_{g,r}^{\otimes}$ considered in [7]. By applying (an argument similar to) the argument in *loc. cit.* to our situation, we will be able to verify the non-projectedness (and hence, the non-splitness) of (the underlying superstack of) ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ even when S_0 is not necessarily $\text{Spec}(\mathbb{C})$ (where \mathbb{C} denotes the field of complex numbers). Indeed, since the non-projectedness of ${}^{\S_1}\mathfrak{M}_{g,r}^{\otimes}$ implies the non-projectedness of ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$, it follows from Theorems 1.1-1.3 in *loc. cit.* that ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ is non-projected for many cases of (g, r) . This means that ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ cannot be reconstructed from purely bosonic moduli stacks in an elementally fashion and, in some sense, needs to be studied independently.

0.5. Let us briefly explain the points of our discussion and the organization of the present paper. In § 1, we give (and recall) a general framework of a theory of superschemes, and more generally, superstacks. Then, we define, in

§2, a logarithmic structure on a superscheme, as well as a superstack. The motivation for introducing the notion of a log superstack (i.e., a superstack equipped with a logarithmic structure) is to consider a supersymmetric analogue of (stable) pointed twisted curves with a canonical log structure. (We refer to [3], Definition 4.3.1, and [5], Definition 2.4.1, for the definition of a pointed twisted curve, and to [19], Theorem 3.5, for the canonical logarithmic structure defined on a pointed twisted curve.) By means of the various notions defined in §§1-2, we present, in §3, the definition of a (stable) pointed log twisted SUSY₁ curve as, roughly speaking, a certain pointed log superstack of superdimension 1|1 equipped with an additional superconformal structure (cf. Definition 3.4.1 and Definition 3.6.1). Thus, for a suitable triple (g, r, λ) of non-negative integers, one may obtain the category ${}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda}^{\otimes \log}$ fibered in groupoids, as we introduced above, classifying stable log twisted SUSY₁ curves of type (g, r, λ) . Denote by $({}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda})_t^{\log}$ the restriction of ${}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda}^{\otimes \log}$ to the full subcategory $\mathcal{S}ch_{/S_0}^{\log} \subseteq \mathcal{S}ch_{/S_0}^{\otimes \log}$ consisting of fs log schemes (in the classical sense). As discussed in §4, the key point is that to giving a family of pointed log twisted SUSY₁ curves parametrized by an fs log scheme (i.e., an object in $({}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda})_t^{\log}$) is equivalent to giving a family of pointed log twisted curves equipped with an additional data called a pointed spin structure (cf. Definition 4.1.1). (This observation for the case of unpointed smooth SUSY₁ curves is classical and well-known.) This implies (cf. Proposition 4.4.1) that $({}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda})_t^{\log}$ is canonically isomorphic to the moduli stack ${}^{tw}\overline{\mathcal{M}}_{g,r,\lambda,\text{spin}}^{\log}$ (cf. (124)) classifying λ -stable log twisted curves of type (g, r) equipped with a pointed spin structure. On the other hand, D. Abramovich, T. J. Jarvis, and A. Chiodo proved (cf. [2], Theorem 1.5.1 and [5], Corollary 4.11) that ${}^{tw}\overline{\mathcal{M}}_{g,r,\lambda,\text{spin}}^{\log} (\cong ({}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda})_t^{\log})$ may be represented by a proper smooth Deligne-Mumford stack, which forms a compactification of the moduli stack classifying pointed smooth spin curves (in the classical sense). Thus, by thickening this Deligne-Mumford stack in the fermionic directions in a way that a universal stable log twisted SUSY₁ curve exists (uniquely), we construct, in §5, a log superstack representing ${}^{\S 1}\overline{\mathcal{M}}_{g,r,\lambda}^{\otimes \log}$ and satisfying the desired conditions described in Theorem A.

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1. SUPERSCHEMES AND SUPERSTACKS

The aim of this section is to give a brief introduction to the theory of superschemes (or more generally, superstacks). We first recall the notion of a superscheme (cf. Definition of 1.1.1) and then, discuss basic properties of superschemes and morphisms between them. In particular, we define a superétale morphism (cf. Definition 1.2.4), which is a supersymmetric analogue of an étale morphism in the classical sense. By means of this sort of morphism, one obtains a category of superschemes equipped with the Grothendieck pretopology which will be denoted by $\mathfrak{Sch}_{/S_0}^{\otimes}$ (cf. (12)), and moreover, obtains the definition of a (Deligne-Mumford) superstack (cf. Definitions 1.6.1 and 1.7.1). Finally, we show (cf. Proposition 1.8.3) that any Deligne-Mumford stack admits a complete versal family which is isomorphic to a split (cf. Definition 1.4.1) and supersmooth (cf. Definition 1.5.1) superscheme. Basic references for the notion of a *superscheme* are, e.g., [4], [9], and [17].

Let R_0 be a noetherian ring over $\mathbb{Z}[\frac{1}{2}]$. Throughout the present paper, *all schemes are assumed to be locally noetherian schemes over the affine scheme $S_0 := \text{Spec}(R_0)$ and all morphisms of schemes are assumed to be locally of finite presentation.*

1.1. Superschemes.

First, recall the definition of a superscheme as follows.

Definition 1.1.1.

- (i) A **superscheme** (over S_0) is a pair $X^{\otimes} := (X_b, \mathcal{O}_{X^{\otimes}})$ consisting of a (locally noetherian) scheme X_b over S_0 and a *coherent* sheaf of superalgebras $\mathcal{O}_{X^{\otimes}}$ over \mathcal{O}_{X_b} such that the natural morphism $\mathcal{O}_{X_b} \rightarrow \mathcal{O}_{X^{\otimes}}$ is injective and its image coincides with the bosonic (i.e., even) part of $\mathcal{O}_{X^{\otimes}}$. We write \mathcal{O}_{X_f} for the fermionic (i.e., odd) part of $\mathcal{O}_{X^{\otimes}}$ and identify \mathcal{O}_{X_b} with the bosonic part via the injection $\mathcal{O}_{X_b} \hookrightarrow \mathcal{O}_{X^{\otimes}}$ (hence, $\mathcal{O}_{X^{\otimes}} = \mathcal{O}_{X_b} \oplus \mathcal{O}_{X_f}$).
- (ii) Let $X^{\otimes} := (X_b, \mathcal{O}_{X^{\otimes}})$ and $Y^{\otimes} := (Y_b, \mathcal{O}_{Y^{\otimes}})$ be two superschemes (over S_0). A **morphism of superschemes (over S_0)** from Y^{\otimes} to X^{\otimes} is a pair $f^{\otimes} := (f_b, f^{\flat})$ consisting of a morphism $f_b : Y_b \rightarrow X_b$ of schemes (over S_0 , which is locally of finite presentation) and a morphism of superalgebras $f^{\flat} : f_b^*(\mathcal{O}_{X^{\otimes}}) (= \mathcal{O}_{Y_b} \otimes_{f_b^{-1}(\mathcal{O}_{X_b})} f_b^{-1}(\mathcal{O}_{X^{\otimes}})) \rightarrow \mathcal{O}_{Y^{\otimes}}$ over \mathcal{O}_{Y_b} .

We always identify any scheme X_b (over S_0) with a superscheme $X^{\otimes} := (X_b, \mathcal{O}_{X^{\otimes}})$ with $\mathcal{O}_{X_f} = 0$.

Remark 1.1.2.

The definition of a superscheme may differ from the usual definition in the sense that the fermionic part of the structure sheaf of a superscheme is assumed not to be coherent. In fact, that condition is usually regarded as an additional condition on a superscheme which is, in [9], Definition 2.6, referred to as “*fermionically of finite presentation*”. But, in the present paper, we only deal with superschemes fermionically of finite presentation in order that superétale morphisms, as well as supersmooth morphisms, defined later are well-behaved. This is why we define the notion of a superscheme as above.

Let X^\otimes be a superscheme and \mathcal{F} a left \mathcal{O}_{X^\otimes} -supermodule. We write

$$(2) \quad \mathcal{F}_b \quad (\text{resp.}, \mathcal{F}_f)$$

for the bosonic (resp., fermionic) part of \mathcal{F} (hence $\mathcal{F} = \mathcal{F}_b \oplus \mathcal{F}_f$). \mathcal{F} may be considered as a right \mathcal{O}_{X^\otimes} -supermodule equipped with an \mathcal{O}_{X^\otimes} -action given by $m \cdot a := (-1)^{|m| \cdot |a|} a \cdot m$ for homogeneous local sections $a \in \mathcal{O}_{X^\otimes}$, $m \in \mathcal{F}$ (where $|\cdot|$ denotes the parity function). By an **\mathcal{O}_{X^\otimes} -supermodule**, we shall mean simply a left \mathcal{O}_{X^\otimes} -supermodule, which is often treated as a right \mathcal{O}_{X^\otimes} -supermodule by this consideration. Also, by a **supervector bundle (of superrank $m|n$)** on X^\otimes (where both m and n are nonnegative integers), we mean a locally free (left) \mathcal{O}_{X^\otimes} -supermodule (of superrank $m|n$).

Let $f^\otimes := (f_b, f^b) : Y^\otimes \rightarrow X^\otimes$ be a morphism of superschemes. If we are given an \mathcal{O}_{X^\otimes} -supermodule (resp., an \mathcal{O}_{Y^\otimes} -supermodule) \mathcal{F} , then one may define, via the natural morphism $f_b^{-1}(\mathcal{O}_{X^\otimes}) \rightarrow \mathcal{O}_{Y^\otimes}$, the pull-back (resp., direct image) of \mathcal{F} to be the \mathcal{O}_{Y^\otimes} -supermodule (resp., the \mathcal{O}_{X^\otimes} -supermodule)

$$(3) \quad f^{\otimes*}(\mathcal{F}) := \mathcal{O}_{Y^\otimes} \otimes_{f_b^{-1}(\mathcal{O}_{X^\otimes})} f_b^{-1}(\mathcal{F}) \quad (\text{resp.}, f_*^\otimes(\mathcal{F}) := f_{b*}(\mathcal{F})).$$

Definition 1.1.3.

Let S^\otimes be a superscheme.

- (i) Let X^\otimes and Y^\otimes be superschemes over S^\otimes and $f^\otimes (:= (f_b, f^b)) : Y^\otimes \rightarrow X^\otimes$ a morphism of superschemes over S^\otimes . We shall say that f^\otimes is a **closed immersion (over S^\otimes)** if $f_b : Y_b \rightarrow X_b$ is a closed immersion and $f^b : f_b^*(\mathcal{O}_{X^\otimes}) \rightarrow \mathcal{O}_Y$ is surjective.
- (ii) Let X^\otimes be a superscheme over S^\otimes . A **closed subsuperscheme** of X^\otimes is an equivalence class of closed immersions into X^\otimes , where two morphisms $f_1^\otimes : Y_1^\otimes \rightarrow X^\otimes$, $f_2^\otimes : Y_2^\otimes \rightarrow X^\otimes$ over S^\otimes are **equivalent** if there exists an isomorphism $\iota^\otimes : Y_1^\otimes \xrightarrow{\sim} Y_2^\otimes$ satisfying that $f_2^\otimes \circ \iota^\otimes = f_1^\otimes$. If $f^\otimes : Y^\otimes \rightarrow X^\otimes$ is a closed immersion, then we shall write $[f^\otimes]$ for the closed subsuperscheme of X^\otimes represented by f^\otimes .

Let $X^\circledast := (X_b, \mathcal{O}_{X^\circledast})$ be a superscheme. By means of the morphism

$$(4) \quad \beta_X^\circledast : X^\circledast \rightarrow X_b$$

corresponding to the inclusion $\mathcal{O}_{X_b} \hookrightarrow \mathcal{O}_{X^\circledast}$, X^\circledast may be thought of as a superscheme over the scheme X_b . The construction of β_X^\circledast is evidently functorial in X^\circledast , that is to say, $\beta_X^\circledast \circ f^\circledast = f_b \circ \beta_Y^\circledast$ for any superscheme Y^\circledast and any morphism $f^\circledast := (f_b, f^b) : Y^\circledast \rightarrow X^\circledast$.

Denote by

$$(5) \quad \mathcal{N}_{X^\circledast}$$

the superideal of $\mathcal{O}_{X^\circledast}$ generated by \mathcal{O}_{X_f} . We shall write

$$(6) \quad \tau_X^\circledast : X_t \rightarrow X^\circledast$$

for the closed immersion corresponding to the quotient $\mathcal{O}_{X^\circledast} \twoheadrightarrow \mathcal{O}_{X^\circledast}/\mathcal{N}_{X^\circledast}$. Hence, X_t forms a scheme, and the composite

$$(7) \quad \gamma_X := \beta_X^\circledast \circ \tau_X^\circledast : X_t \rightarrow X_b$$

forms a closed immersion of schemes corresponding to the quotient $\mathcal{O}_{X_b} \twoheadrightarrow \mathcal{O}_{X_b}/\mathcal{O}_{X_f}^2 (= \mathcal{O}_{X^\circledast}/\mathcal{N}_{X^\circledast})$ by the nilpotent ideal $\mathcal{O}_{X_f}^2 \subseteq \mathcal{O}_{X_b}$. Any morphism $f^\circledast : Y^\circledast \rightarrow X^\circledast$ induces a morphism $f_t : Y_t \rightarrow X_t$ of schemes satisfying that $f^\circledast \circ \tau_Y^\circledast = \tau_X^\circledast \circ f_t$ and $f_b \circ \tau_Y = \tau_X \circ f_t$. In particular, any morphism $Z \rightarrow X^\circledast$

(where Z is a scheme) decomposes as $Z \rightarrow X_t \xrightarrow{\tau_X^\circledast} X^\circledast$ for a unique morphism $Z \rightarrow X_t$ of schemes.

Finally, for each nonnegative integer n , we write

$$(8) \quad \mathrm{gr}_{X^\circledast}^n := \mathcal{N}_{X^\circledast}^n / \mathcal{N}_{X^\circledast}^{n+1},$$

which may be thought of as an \mathcal{O}_{X_t} -module.

1.2. Morphisms of superschemes.

We shall consider analogues of flat morphism and étale morphisms to superschemes. Let $f^\circledast := (f_b, f^b) : Y^\circledast \rightarrow X^\circledast$ be a morphism of superschemes.

Definition 1.2.1.

We shall say that f^\circledast is **bosonic** if for any scheme Z together with a morphism $Z \rightarrow X^\circledast$, the fiber product $Y^\circledast \times_{f^\circledast, X^\circledast} Z$ is a scheme. (Here, we note that the superschemes and morphisms between them form a category, in which the fiber products exist. See [4], Corollary 10.3.9.)

Definition 1.2.2.

We shall say that f^\circledast is **superflat** if for any point y of Y_b the homomorphism $\mathcal{O}_{X^\circledast, f_b(y)} \rightarrow \mathcal{O}_{Y^\circledast, y}$ of local rings induced by f^\circledast is flat.

Remark 1.2.3.

Suppose that f^\otimes is bosonic and superflat. According to [9], Lemma 2.7 and Proposition 2.1 (cf. Remark 1.1.2), the following properties hold (where although the results of *loc. cit.* is assumed that $S_0 = \text{Spec}(\mathbb{C})$, one may prove the same assertion for our general case):

(i) The homomorphism f^b induces, by restriction, isomorphisms

$$(9) \quad f_b^*(\mathcal{O}_{X_f}) \xrightarrow{\sim} \mathcal{O}_{Y_f} \quad \text{and} \quad f_b^*(\mathcal{O}_{X_f}^2) \xrightarrow{\sim} \mathcal{O}_{Y_f}^2$$

(hence, we have $f_b^*(\mathcal{N}_{X^\otimes}) \xrightarrow{\sim} \mathcal{N}_{Y^\otimes}$). In particular, the natural morphisms

$$(10) \quad Y^\otimes \rightarrow Y_b \times_{f_b, X_b, \beta_X^\otimes} X^\otimes \quad \text{and} \quad Y_t \rightarrow Y^\otimes \times_{f^\otimes, X^\otimes, \tau_X^\otimes} X_t$$

are isomorphisms.

(ii) The underlying morphism $f_b : Y_b \rightarrow X_b$ is flat (in the classical sense).

Definition 1.2.4.

We shall say that f^\otimes is **superétale** if f^\otimes is bosonic and superflat, and the flat morphism $f_b : Y_b \rightarrow X_b$ (cf. Remark 1.2.3 (ii) above) is unramified.

Proposition 1.2.5.

For a superscheme Z^\otimes over S_0 , we shall denote by $\mathfrak{Et}_{/Z^\otimes}$ the category defined as follows:

- The objects are superétale morphisms $W^\otimes \rightarrow Z^\otimes$ of superschemes to Z^\otimes ;
- The morphisms from $W_1^\otimes \rightarrow Z^\otimes$ to $W_2^\otimes \rightarrow Z^\otimes$ (where both $W_1^\otimes \rightarrow Z^\otimes$ and $W_2^\otimes \rightarrow Z^\otimes$ are objects of this category) are morphisms $W_1^\otimes \rightarrow W_2^\otimes$ of superschemes over Z^\otimes .

Then, the functor

$$(11) \quad \mathfrak{Et}_{/X^\otimes} \xrightarrow{\sim} \mathfrak{Et}_{/X_t}.$$

determined by base-change $Y^\otimes \mapsto Y^\otimes \times_{X^\otimes, \tau_X^\otimes} X_t$ is an equivalence of categories.

In particular, if X'^\otimes and X''^\otimes are superschemes over S_0 such that $(X'_t)_{\text{red}} \cong (X''_t)_{\text{red}}$ (where $(-)_{\text{red}}$ denotes the reduced scheme associated with the scheme $(-)$), then we have $\mathfrak{Et}_{/X'^\otimes} \cong \mathfrak{Et}_{/X''^\otimes}$.

Proof. We shall construct a functor $\mathfrak{Et}_{/X_t} \rightarrow \mathfrak{Et}_{/X^\otimes}$. Let $Y_0 \rightarrow X_t$ be an object in $\mathfrak{Et}_{/X_t}$ (i.e., $Y_0 \rightarrow X_t$ is étale in the classical sense). Since X_b is a nilpotent thickening of X_t (via the closed immersion γ_X), Y_0 extends *uniquely* to an étale scheme Y_1 over X_b . The superscheme $Y_1 \times_{X_b} X^\otimes$ (together with the projection to X^\otimes) is an object of $\mathfrak{Et}_{/X^\otimes}$ whose image of the functor (11) is isomorphic to Y_0 . The assignment $Y_0 \mapsto Y_1 \times_{X_b} X^\otimes$ is well-defined and functorial with

respect to Y_0 , and hence, determines a functor $\mathfrak{Et}_{/X_t} \rightarrow \mathfrak{Et}_{/X^\otimes}$. This functor is verified to be the inverse to the functor (11). This completes the proof of Proposition 1.2.5. \square

1.3. The category of superschemes.

Write

$$(12) \quad \mathfrak{Sch}_{/S_0} \text{ (resp., } \mathfrak{Sch}_{/S_0}^{\otimes})$$

for the category whose *objects* are schemes (resp., superschemes) and whose *morphisms* are morphisms of schemes (resp., morphisms of superschemes). By the natural inclusion $\mathfrak{Sch}_{/S_0} \hookrightarrow \mathfrak{Sch}_{/S_0}^{\otimes}$, we identify $\mathfrak{Sch}_{/S_0}$ with a full subcategory of $\mathfrak{Sch}_{/S_0}^{\otimes}$. The fiber products and finite coproducts exist in $\mathfrak{Sch}_{/S_0}^{\otimes}$, and the inclusion $\mathfrak{Sch}_{/S_0} \hookrightarrow \mathfrak{Sch}_{/S_0}^{\otimes}$ preserves the fiber products and finite coproducts. When there is fear of confusion, by a **stack (over S_0)**, we mean a stack over the site $\mathfrak{Sch}_{/S_0}$ with respect to the étale pretopology. We shall equip $\mathfrak{Sch}_{/S_0}^{\otimes}$ with the Grothendieck pretopology consisting of coverings $\{U_i^{\otimes} \rightarrow X^{\otimes}\}_{i \in I}$, where each $U_i^{\otimes} \rightarrow X^{\otimes}$ is a superétale morphism such that (the underlying morphism between schemes of) $\coprod_{i \in I} U_i^{\otimes} \rightarrow X^{\otimes}$ is surjective; we shall refer to this pretopology as the **superétale pretopology**. One verifies that the property on a morphism in $\mathfrak{Sch}_{/S_0}^{\otimes}$ of being bosonic (resp., superflat; resp., superétale) is closed under composition and base-change, and satisfies descent for superétale coverings.

1.4. Split superschemes and affine superschemes.

Let \underline{X} be a scheme and \mathcal{E} a coherent $\mathcal{O}_{\underline{X}}$ -module. Consider the exterior algebra $\bigwedge_{\mathcal{O}_{\underline{X}}}^{\bullet} \mathcal{E} := \bigoplus_{i \geq 0} \bigwedge_{\mathcal{O}_{\underline{X}}}^i \mathcal{E}$ associated with \mathcal{E} over $\mathcal{O}_{\underline{X}}$. The $\mathcal{O}_{\underline{X}}$ -subalgebra $\bigwedge_{\mathcal{O}_{\underline{X}}}^{\text{even}} \mathcal{E} := \bigoplus_{i: \text{even}} \bigwedge_{\mathcal{O}_{\underline{X}}}^i \mathcal{E}$ defines (since it is commutative) a relative affine space $\text{Spec}(\bigwedge_{\mathcal{O}_{\underline{X}}}^{\text{even}} \mathcal{E})$ over \underline{X} . Also, $\bigwedge_{\mathcal{O}_{\underline{X}}}^{\bullet} \mathcal{E}$ may be thought of as a coherent $\mathcal{O}_{\text{Spec}(\bigwedge_{\mathcal{O}_{\underline{X}}}^{\text{even}} \mathcal{E})}$ -module. Thus, we obtain a superscheme

$$(13) \quad \langle \underline{X}, \mathcal{E} \rangle^{\otimes} := (\text{Spec}(\bigwedge_{\mathcal{O}_{\underline{X}}}^{\text{even}} \mathcal{E}), \bigwedge_{\mathcal{O}_{\underline{X}}}^{\bullet} \mathcal{E}).$$

The inclusion $\mathcal{O}_{\underline{X}} (= \bigwedge_{\mathcal{O}_{\underline{X}}}^0 \mathcal{E}) \hookrightarrow \bigwedge_{\mathcal{O}_{\underline{X}}}^{\bullet} \mathcal{E}$ defines a morphism

$$(14) \quad \langle \beta \rangle_{\underline{X}, \mathcal{E}}^{\otimes} : \langle \underline{X}, \mathcal{E} \rangle^{\otimes} \rightarrow \underline{X}$$

of superschemes.

Definition 1.4.1.

We shall say that a superscheme Z^{\otimes} is **split** if $Z^{\otimes} \cong \langle \underline{X}, \mathcal{E} \rangle^{\otimes}$ for some scheme \underline{X} and a coherent $\mathcal{O}_{\underline{X}}$ -module \mathcal{E} .

Next, we shall recall the notion of an affine superscheme.

Definition 1.4.2.

If $R := R_b \oplus R_f$ is a superring, then we shall write

$$(15) \quad \mathrm{Spec}(R)^{\otimes}$$

for the superspectrum of R (i.e., “ $\mathrm{Spec} R$ ” in the sense of [4], Definition 10.1.1). We shall say that a superscheme \overline{X}^{\otimes} is **affine** if it is isomorphic to $\mathrm{Spec}(R)^{\otimes}$ for some superring R (in particular, both X_b and X_f are affine schemes).

If both m and n are nonnegative integers and S^{\otimes} is a superscheme, then we shall write

$$(16) \quad \mathbb{A}_{S^{\otimes}}^{m|n} := S^{\otimes} \times_{S_0} \mathrm{Spec}(R_0[t_n, \dots, t_m] \otimes_{R_0} \bigwedge_{R_0} (R_0\psi_1 \oplus \dots \oplus R_0\psi_n))^{\otimes}$$

$$\left(= S^{\otimes} \times_{S_0} \langle \mathbb{A}_{S_0}^{m|0}, \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{A}_{S_0}^{m|0}} \psi_l \rangle^{\otimes} \right),$$

where the t_1, \dots, t_m are ordinary indeterminates and ψ_1, \dots, ψ_n are fermionic (i.e., anticommuting) indeterminates.

The following assertion is immediately verified from the definition of $\mathbb{A}_{(-)}^{m|n}$.

Proposition 1.4.3.

Let $f^{\otimes} : Y^{\otimes} \rightarrow X^{\otimes}$ be a morphism of superschemes. Then, the functorial (with respect to Y^{\otimes}) map of sets

$$(17) \quad \{\tilde{f}^{\otimes} \in \mathrm{Hom}_{\mathfrak{Sct}/S_0}(Y^{\otimes}, \mathbb{A}_{X^{\otimes}}^{m|n}) \mid \mathrm{pr} \circ \tilde{f}^{\otimes} = f^{\otimes}\} \rightarrow \Gamma(Y_b, \mathcal{O}_{Y_b})^{\oplus m} \oplus \Gamma(Y_f, \mathcal{O}_{Y_f})^{\oplus n}$$

$$\tilde{f}^{\otimes} \mapsto ((\tilde{f}^b(t_l))_{l=1}^m, (\tilde{f}^f(\psi_{l'})_{l'=1}^n)$$

is bijective, where pr denotes the natural projection $\mathbb{A}_{X^{\otimes}}^{m|n} \rightarrow X^{\otimes}$.

1.5. Supersmooth morphisms.

Let m and n be nonnegative integers.

Definition 1.5.1.

Let $f^{\otimes} : Y^{\otimes} \rightarrow X^{\otimes}$ be a morphism of superschemes. We shall say that f^{\otimes} is **supersmooth of relative superdimension $m|n$** if there exists, superétale locally on Y^{\otimes} , a superétale morphism $Y^{\otimes} \rightarrow X^{\otimes} \times_{S_0} \mathbb{A}_{S_0}^{m|n}$ over X^{\otimes} .

Remark 1.5.2.

Let n be a nonnegative integer. A morphism $f^\otimes : Y^\otimes \rightarrow X^\otimes$ of superschemes is supersmooth of relative superdimension $n|0$ if and only if f^\otimes is superflat and $f_b : Y_b \rightarrow X_b$ is, in the classical sense, smooth of relative dimension n (i.e., all nonempty fibers are equidimensional of dimension n). In particular, f^\otimes is supersmooth of relative superdimension $0|0$ if and only if it is superétale.

Proposition 1.5.3.

Let X^\otimes be a superscheme. Then, the following two conditions (a) and (b) are equivalent:

- (a) X^\otimes is supersmooth over S_0 of relative superdimension $m|n$;
- (b) X_t is smooth over S_0 of relative dimension m , the \mathcal{O}_{X_t} -module $\mathrm{gr}_{X^\otimes}^1$ is locally free of rank n , and there exists, superétale locally on X^\otimes , an isomorphism $X^\otimes \xrightarrow{\sim} \langle X_t, \mathrm{gr}_{X^\otimes}^1 \rangle$ which makes the diagram

$$(18) \quad \begin{array}{ccc} & X_t & \\ \tau_X^\otimes \swarrow & & \searrow \tau_{\langle X_t, \mathrm{gr}_{X^\otimes}^1 \rangle}^\otimes \\ X^\otimes & \xrightarrow{\sim} & \langle X_t, \mathrm{gr}_{X^\otimes}^1 \rangle \end{array}$$

commute.

In particular, X^\otimes is split and supersmooth over S_0 of relative superdimension $m|n$ if and only if $X^\otimes \cong \langle \underline{X}, \mathcal{E} \rangle^\otimes$ for some smooth scheme \underline{X} over S_0 of relative dimension m and some vector bundle \mathcal{E} on \underline{X} of rank n .

Proof. Since the latter assertion follows directly from the former assertion, it suffices to prove only the former assertion, i.e., the equivalence (a) \Leftrightarrow (b). The implication (b) \Rightarrow (a) is clear. We shall prove (a) \Rightarrow (b). After possibly replacing X^\otimes with its superétale covering, we may assume, without loss of generality, that X_b is affine and there exists a (globally defined) superétale morphism $\pi^\otimes (:= (\pi_b, \pi^b)) : X^\otimes \rightarrow \mathbb{A}_{S_0}^{m|n}$ over S_0 . Then, π^b restricts to isomorphisms

$$(19) \quad \pi_b^*(\mathcal{O}_{(\mathbb{A}_{S_0}^{m|n})_f}^i) \cong \mathcal{O}_{X_f}^i \quad (i = 1, 2)$$

(cf. Remark 1.2.3). In particular, the commutative square diagram

$$(20) \quad \begin{array}{ccc} X_t & \xrightarrow{\gamma_X} & X_b \\ \pi_t \downarrow & & \downarrow \pi_b \\ (\mathbb{A}_{S_0}^{m|n})_t & \xrightarrow{\gamma_{\mathbb{A}_{S_0}^{m|n}}} & (\mathbb{A}_{S_0}^{m|n})_b \end{array}$$

is cartesian. It follows that X_t is étale over $(\mathbb{A}_{S_0}^{m|n})_t (= \mathbb{A}_{S_0}^{m|0})$, and hence, smooth over S_0 of relative dimension m . Since X_b is affine, there exists a morphism

$\iota_X : X_b \rightarrow X_t$ over S_0 satisfying that $\iota_X \circ \gamma_X = \text{id}_{X_t}$. The isomorphisms (19) yield an isomorphism

$$(21) \quad \text{gr}_{\pi^b}^1 : (\pi_t^*(\text{gr}_{\mathbb{A}_{S_0}^{m|n}}^1) =) \pi_b^*(\text{gr}_{\mathbb{A}_{S_0}^{m|n}}^1) \xrightarrow{\sim} \text{gr}_{X^\otimes}^1.$$

In particular, we have $\text{gr}_{X^\otimes}^1 \cong \mathcal{O}_{X_t}^{\oplus n}$. By Proposition 1.4.3, one may find a morphism

$$(22) \quad \tilde{\iota}_X^\otimes : X^\otimes \rightarrow \langle X_t, \text{gr}_{X^\otimes}^1 \rangle^\otimes$$

whose composite with the projection $\langle \beta \rangle_{X_t, \text{gr}_{X^\otimes}^1}^\otimes : \langle X_t, \text{gr}_{X^\otimes}^1 \rangle^\otimes \rightarrow X_t$ coincides with $\iota_X \circ \beta_X^\otimes : X^\otimes \rightarrow X_t$ and which makes the diagram

$$(23) \quad \begin{array}{ccc} X & \xrightarrow{\tilde{\iota}_X^\otimes} & \langle X_t, \text{gr}_{X^\otimes}^1 \rangle^\otimes \\ & \searrow \pi^\otimes & \swarrow \pi'^\otimes \\ & \mathbb{A}_{S_0}^{m|n} & \end{array}$$

commute, where π'^\otimes denotes the morphism determined (by means of the canonical isomorphism $\mathbb{A}_{S_0}^{m|n} \xrightarrow{\sim} \langle \mathbb{A}_{S_0}^{m|0}, \pi_t^*(\text{gr}_{\mathbb{A}_{S_0}^{m|n}}^1) \rangle^\otimes$) by both $\pi_t : X_t \rightarrow \mathbb{A}_{S_0}^{m|0}$ and the morphism $\text{gr}_{\pi^b}^1$. Observe that both π^\otimes and π'^\otimes are superflat (since $\langle X_t, \text{gr}_{X^\otimes}^1 \rangle^\otimes \cong \mathbb{A}_{S_0}^{m|n} \times_{\mathbb{A}_{S_0}^{m|0}, \pi_t} X_t$ and π_t is étale). On the other hand, $\tilde{\iota}_X$ restricts, via base-change by $\tau_{\mathbb{A}_{S_0}^{m|n}}^\otimes : (\mathbb{A}_{S_0}^{m|n})_t \rightarrow \mathbb{A}_{S_0}^{m|n}$, to the identity morphism of X_t . This implies that $\tilde{\iota}_X$ is an isomorphism, and hence, completes the proof of Proposition 1.5.3. \square

1.6. Superstacks.

Definition 1.6.1.

- (i) A **superstack (over S_0)** is a category fibered in groupoids $Z^\otimes \rightarrow \mathfrak{Sch}_{/S_0}^\otimes$ over $\mathfrak{Sch}_{/S_0}^\otimes$ which is a stack with respect to the superétale pretopology.
- (ii) Let $Z_1^\otimes \rightarrow \mathfrak{Sch}_{/S_0}^\otimes$ and $Z_2^\otimes \rightarrow \mathfrak{Sch}_{/S_0}^\otimes$ be superstacks. A **morphism of superstacks** from Z_1^\otimes to Z_2^\otimes is a functor $Z_1^\otimes \rightarrow Z_2^\otimes$ over $\mathfrak{Sch}_{/S_0}^\otimes$.

One verifies immediately that the superstacks and the morphisms of superstacks form a 2-category, in which the 2-fiber products and finite coproducts exist. The natural inclusion from $\mathfrak{Sch}_{/S_0}^\otimes$ into this category preserves such limits.

Remark 1.6.2.

For a superscheme X^\otimes , the set-valued contravariant functor $\text{Hom}_{\mathfrak{Sch}_{/S_0}^\otimes}(-, X^\otimes)$

on $\mathfrak{Sch}_{/S_0}^{\otimes}$ is verified (from a standard argument in descent theory) to be a sheaf on $\mathfrak{Sch}_{/S_0}^{\otimes}$ with respect to superétale pretopology (cf. [9], Lemma 2.8). We always identify any superscheme X^{\otimes} with the superstack corresponding to the sheaf $\mathrm{Hom}_{\mathfrak{Sch}_{/S_0}^{\otimes}}(-, X^{\otimes})$.

Remark 1.6.3.

If \underline{Z} is a stack over S_0 , then, in a natural manner, one may consider it as a superstack. More precisely, let us define the category $\underline{Z}^{\otimes\text{-triv}} \rightarrow \mathfrak{Sch}_{/S_0}^{\otimes}$ fibered in groupoids over $\mathfrak{Sch}_{/S_0}^{\otimes}$ as follows:

- The *objects* in $\underline{Z}^{\otimes\text{-triv}}$ are pairs (X^{\otimes}, x) consisting of a superscheme X^{\otimes} and a morphism $x : X_b \rightarrow \underline{Z}$ of stacks;
- The *morphisms* from an object (Y^{\otimes}, y) to an object (X^{\otimes}, x) are morphisms $f^{\otimes} : Y^{\otimes} \rightarrow X^{\otimes}$ of superschemes satisfying that $x \circ f_b \cong y$;
- The *functor* $\underline{Z}^{\otimes\text{-triv}} \rightarrow \mathfrak{Sch}_{/S_0}^{\otimes}$ is given by assigning $(X^{\otimes}, x) \mapsto X^{\otimes}$ (for any object (X^{\otimes}, x) in $\underline{Z}^{\otimes\text{-triv}}$) and $f^{\otimes} \mapsto f^{\otimes}$ (for any morphism f^{\otimes} in $\underline{Z}^{\otimes\text{-triv}}$).

Then, $\underline{Z}^{\otimes\text{-triv}}$ forms a superstack. The assignment $\underline{Z} \mapsto \underline{Z}^{\otimes\text{-triv}}$ determines a fully faithful functor from the category of stacks over S_0 to the category of superstacks over S_0 . In this manner, we always consider any stack as a superstack.

Remark 1.6.4.

Let $Z^{\otimes} \rightarrow \mathfrak{Sch}_{/S_0}^{\otimes}$ be a superstack. The restriction of this superstack to the subcategory $\mathfrak{Sch}_{/S_0}^{\otimes} \subseteq \mathfrak{Sch}_{/S_0}^{\otimes}$ forms a stack

$$(24) \quad Z_t \rightarrow \mathfrak{Sch}_{/S_0}$$

over S_0 . If, moreover, Z^{\otimes} may be represented by a superscheme X^{\otimes} (i.e., $Z^{\otimes} \cong \mathrm{Hom}_{\mathfrak{Sch}_{/S_0}^{\otimes}}(-, X^{\otimes})$), then Z_t (in the sense of (24)) may be represented by X_t (in the sense of (6)). Moreover, if $W^{\otimes} \rightarrow Z^{\otimes}$ is a morphism of superstacks, then it induces a morphism $W_t \rightarrow Z_t$ of stacks.

Definition 1.6.5.

Let $f^{\otimes} : Y^{\otimes} \rightarrow X^{\otimes}$ be a morphism of superstacks.

- We shall say that f^{\otimes} is **representable** if, for any morphism $X'^{\otimes} \rightarrow X^{\otimes}$ of superstacks (where X'^{\otimes} is a superscheme), the fiber product $Y^{\otimes} \times_{f^{\otimes}, X^{\otimes}} X'^{\otimes}$ is a superscheme.
- We shall say that f^{\otimes} is **superproper** if the underlying morphism $f_t : Y_t \rightarrow X_t$ of stacks is proper in the classical sense.

1.7. Deligne-Mumford superstacks.

Definition 1.7.1.

We shall say that a superstack Z^\otimes is **Deligne-Mumford** if it satisfies the following two conditions:

- (i) The diagonal morphism $Z^\otimes \rightarrow Z^\otimes \times_{S_0} Z^\otimes$ is representable and the associated (representable) morphism $Z_t \rightarrow Z_t \times_{S_0} Z_t$ of stacks is separated and quasi-compact in the classical sense;
- (ii) There exists a superscheme U^\otimes over S_0 together with a representable morphism $U^\otimes \rightarrow Z^\otimes$ of superstacks over S_0 such that for each superscheme V^\otimes over Z^\otimes , the morphism $U^\otimes \times_{Z^\otimes} V^\otimes \rightarrow V^\otimes$ of superschemes (where $U^\otimes \times_{Z^\otimes} V^\otimes$ is necessarily a superscheme thanks to condition (i)) is surjective and superétale.

We shall refer to such a superscheme U^\otimes (together with $U^\otimes \rightarrow Z^\otimes$) as a **complete versal family for Z^\otimes** .

Definition 1.7.2.

Let $f^\otimes : Y^\otimes \rightarrow X^\otimes$ be a morphism of Deligne-Mumford superstacks over S_0 , and let m, n be nonnegative integers. We shall say that f^\otimes is **superétale** (resp., **supersmooth of relative superdimension $m|n$**) if for any 2-commutative diagram

$$(25) \quad \begin{array}{ccccc} V^\otimes & \longrightarrow & Y'^\otimes & \longrightarrow & Y^\otimes \\ & \searrow h^\otimes & \downarrow & \square & \downarrow f^\otimes \\ & & U^\otimes & \longrightarrow & X^\otimes, \end{array}$$

where U^\otimes and V^\otimes are complete versal families for X^\otimes and $Y'^\otimes := Y^\otimes \times_{X^\otimes} U^\otimes$ respectively, the morphism $h^\otimes : V^\otimes \rightarrow U^\otimes$ of superschemes is superétale in the sense of Definition 1.2.4 (resp., supersmooth of relative superdimension $m|n$ in the sense of Definition 1.5.1).

Remark 1.7.3.

Let Z^\otimes be a Deligne-Mumford superstack. Then, the structure sheaf \mathcal{O}_{Z^\otimes} on Z^\otimes is defined to be a superétale sheaf on Z^\otimes such that $\Gamma(T^\otimes, \mathcal{O}_{Z^\otimes}) := \Gamma(T_b, \mathcal{O}_{T^\otimes})$ for any superscheme T^\otimes together with a superétale morphism $T^\otimes \rightarrow Z^\otimes$. Moreover, one may define the notion of an \mathcal{O}_{Z^\otimes} -supermodule, as usual (cf. the discussion following Remark 1.1.2).

1.8. Groupoids in the category of superschemes.

Now, we recall that if we are given a *groupoid* Γ , then it may be described as a certain collection of data $(U_0, R_0, s_0, t_0, c_0)$, where U_0 and R_0 denote the sets of *objects* and *arrows* of Γ respectively, s_0 and t_0 denote the *source* and *target* maps

$R_0 \rightarrow U_0$ respectively, and c_0 denotes the *composition* map $R_0 \times_{t_0, U_0, s_0} R_0 \rightarrow R_0$.

Definition 1.8.1.

A **groupoid in $\mathfrak{Sch}_{/S_0}^{\otimes}$** is a collection of data

$$(26) \quad R^{\otimes} \rightrightarrows U^{\otimes} := (U^{\otimes}, R^{\otimes}, s^{\otimes}, t^{\otimes}, c^{\otimes}),$$

where

- U^{\otimes} and R^{\otimes} are superschemes;
- $s^{\otimes}, t^{\otimes} : R^{\otimes} \rightarrow U^{\otimes}$ and $c^{\otimes} : R^{\otimes} \times_{s^{\otimes}, U^{\otimes}, t^{\otimes}} R^{\otimes} \rightarrow R^{\otimes}$ are morphisms of superschemes

such that for any $T^{\otimes} \in \text{Ob}(\mathfrak{Sch}_{/S_0}^{\otimes})$ the quintuple

$$(27) \quad (R^{\otimes} \rightrightarrows U^{\otimes})(T^{\otimes}) := (U^{\otimes}(T^{\otimes}), R^{\otimes}(T^{\otimes}), s^{\otimes}(T^{\otimes}), t^{\otimes}(T^{\otimes}), c^{\otimes}(T^{\otimes}))$$

forms a groupoid (in the above sense) which is functorial with respect to T^{\otimes} . In a similar vein, one may obtain the definition of a **groupoid in $\mathfrak{Sch}_{/S_0}$** .

As in the usual case of stacks, one may associate, to each groupoid $R^{\otimes} \rightrightarrows U^{\otimes}$ in $\mathfrak{Sch}_{/S_0}^{\otimes}$, a superstack

$$(28) \quad [R^{\otimes} \rightrightarrows U^{\otimes}]$$

over S_0 . More precisely, $[R^{\otimes} \rightrightarrows U^{\otimes}]$ is the *stackification* (with respect to the superétale pretopology) of the category fibered in groupoids $[R^{\otimes} \rightrightarrows U^{\otimes}]'$ determined by $R^{\otimes} \rightrightarrows U^{\otimes}$ (i.e., the fiber of $[R^{\otimes} \rightrightarrows U^{\otimes}]'$ over $T^{\otimes} \in \text{Ob}(\mathfrak{Sch}_{/S_0}^{\otimes})$ is the groupoid $(R^{\otimes} \rightrightarrows U^{\otimes})(T^{\otimes})$ defined above). Denote by

$$(29) \quad \pi_{R^{\otimes} \rightrightarrows U^{\otimes}}^{\otimes} : U^{\otimes} \rightarrow [R^{\otimes} \rightrightarrows U^{\otimes}]$$

the natural projection.

Remark 1.8.2.

Let Z^{\otimes} be a Deligne-Mumford superstack.

- (i) One verifies that there exists an isomorphism $z^{\otimes} : [R^{\otimes} \rightrightarrows U^{\otimes}] \xrightarrow{\sim} Z^{\otimes}$ of superstacks, where $R^{\otimes} \rightrightarrows U^{\otimes} := (U^{\otimes}, R^{\otimes}, s^{\otimes}, t^{\otimes}, c^{\otimes})$ is a groupoid in $\mathfrak{Sch}_{/S_0}^{\otimes}$, such that the three morphisms s^{\otimes} , t^{\otimes} , and $z^{\otimes} \circ \pi_{R^{\otimes} \rightrightarrows U^{\otimes}}^{\otimes}$ are superétale. Indeed, if U^{\otimes} is an arbitrary complete versal family for our Z^{\otimes} , then one may obtain, by starting with the data U^{\otimes} , the desired groupoid $[R^{\otimes} \rightrightarrows U^{\otimes}]$ in $\mathfrak{Sch}_{/S_0}^{\otimes}$ as follows:

- $R^{\otimes} := U^{\otimes} \times_{Z^{\otimes}} U^{\otimes}$;
- s^{\otimes} and t^{\otimes} are the second and first projections $U^{\otimes} \times_{Z^{\otimes}} U^{\otimes} (= R^{\otimes}) \rightarrow U^{\otimes}$ respectively;

- $c^\mathbb{S}$ is the projection

$$(30) \quad (U^\mathbb{S} \times_{Z^\mathbb{S}} U^\mathbb{S}) \times_{s^\mathbb{S}, U^\mathbb{S}, t^\mathbb{S}} (U^\mathbb{S} \times_{Z^\mathbb{S}} U^\mathbb{S}) (= R^\mathbb{S} \times_{s^\mathbb{S}, U^\mathbb{S}, t^\mathbb{S}} R^\mathbb{S}) \\ \rightarrow U^\mathbb{S} \times_{Z^\mathbb{S}} U^\mathbb{S} (= R^\mathbb{S})$$

into the (1, 4)-th factor.

We shall refer to such a groupoid $R^\mathbb{S} \rightrightarrows U^\mathbb{S}$ (together with such an isomorphism $z^\mathbb{S}$) as a **representation** of $Z^\mathbb{S}$.

- (ii) Let $R^\mathbb{S} \rightrightarrows U^\mathbb{S}$ be as in (i) and denote by $R_t \rightrightarrows U_t$ the groupoid in $\mathfrak{Sch}_{/S_0}$ defined to be $R_t \rightrightarrows U_t := (U_t, R_t, s_t, t_t, c_t)$. Then, the isomorphism $z^\mathbb{S}$ induces an isomorphism

$$(31) \quad z_t : [R_t \rightrightarrows U_t] \xrightarrow{\sim} Z_t$$

of stacks.

- (iii) Let $R^\mathbb{S} \rightrightarrows U^\mathbb{S}$ be as in (i) again. Then, $R_b \rightrightarrows U_b := (U_b, R_b, s_b, t_b, c_b)$ forms a groupoid in $\mathfrak{Sch}_{/S_0}$ and we obtain a stack

$$(32) \quad Z_b := [R_b \rightrightarrows U_b]$$

together with a morphism $\beta_Z^\mathbb{S} : Z^\mathbb{S} \rightarrow Z_b$. For any superétale morphism $T^\mathbb{S} \rightarrow Z^\mathbb{S}$ (where $T^\mathbb{S}$ is a superscheme), there exists an étale morphism $T_b \rightarrow Z_b$ which makes the square diagram

$$(33) \quad \begin{array}{ccc} T^\mathbb{S} & \longrightarrow & Z^\mathbb{S} \\ \beta_T^\mathbb{S} \downarrow & & \downarrow \beta_Z^\mathbb{S} \\ T_b & \longrightarrow & Z_b \end{array}$$

commute and cartesian. In particular, the structure sheaf \mathcal{O}_{Z_b} of Z_b may be identified, via $\beta_Z^\mathbb{S}$, with the bosonic part of \mathcal{O}_Z . If, moreover, $Z^\mathbb{S}$ may be represented by a superscheme $X^\mathbb{S} := (X_b, \mathcal{O}_{X^\mathbb{S}})$, then Z_b and $\beta_Z^\mathbb{S}$ (in the sense of (32)) are isomorphic to X_b and $\beta_X^\mathbb{S}$ (in the sense of (4)) respectively.

Proposition 1.8.3.

Let $Z^\mathbb{S}$ be a superstack and m, n are nonnegative integers. Then, the following three conditions (a), (b), and (c) are equivalent:

- (a) $Z^\mathbb{S}$ is a supersmooth Deligne-Mumford superstack over S_0 of relative superdimension $m|n$;
- (b) $Z^\mathbb{S}$ is a Deligne-Mumford superstack for which there exists a complete versal family of the form $\langle \underline{U}, \mathcal{E}_U \rangle^\mathbb{S}$, where \underline{U} denotes a smooth locally noetherian scheme over S_0 of relative dimension m and \mathcal{E}_U denotes a vector bundle on \underline{U} of rank n ;

- (c) Z^\otimes admits a representation $R^\otimes \rightrightarrows U^\otimes := (U^\otimes, R^\otimes, s^\otimes, t^\otimes, c^\otimes)$ satisfying the following properties:
- (c-1) $U^\otimes = \langle \underline{U}, \mathcal{E}_{\underline{U}} \rangle^\otimes$, where \underline{U} is a smooth scheme over S_0 of relative dimension m and $\mathcal{E}_{\underline{U}}$ is a rank n vector bundle on \underline{U} ;
 - (c-2) Both s^\otimes and t^\otimes are superétale and the morphism $(s_t, t_t) : R_t \rightarrow \underline{U} \times_{S_0} \underline{U} (= U_t \times_{S_0} U_t)$ is separated and quasi-compact.

Proof. The equivalence (a) \Leftrightarrow (c) follows immediately from Proposition 1.5.3 and the definition of a Deligne-Mumford stack. The implication (b) \Rightarrow (c) is clear. Let us consider (c) \Rightarrow (b). First, we prove that the diagonal morphism $\Delta_Z^\otimes : Z^\otimes \rightarrow Z^\otimes \times_{S_0} Z^\otimes$ is representable. Let V^\otimes be an object in $\mathfrak{Sch}_{/S_0}^\otimes$, and let x^\otimes and $y^\otimes : V^\otimes \rightarrow Z^\otimes$ be morphisms in $\mathfrak{Sch}_{/S_0}^\otimes$. To prove that $Z^\otimes \times_{\Delta_Z^\otimes, Z^\otimes \times_{S_0} Z^\otimes, (x^\otimes, y^\otimes)} V^\otimes$ is in $\mathfrak{Sch}_{/S_0}^\otimes$, one may replace (thanks to descent property in the superétale pretopology) V^\otimes with its superétale covering. Hence, we suppose, without loss of generality, that both x^\otimes and y^\otimes may lift to morphisms $\tilde{x}^\otimes, \tilde{y}^\otimes : V^\otimes \rightarrow U^\otimes$. Then, we have

$$(34) \quad Z^\otimes \times_{\Delta_Z^\otimes, Z^\otimes \times_{S_0} Z^\otimes, (x^\otimes, y^\otimes)} V^\otimes \xrightarrow{\sim} R^\otimes \times_{(s^\otimes, t^\otimes), U^\otimes \times_{S_0} U^\otimes, (\tilde{x}^\otimes, \tilde{y}^\otimes)} V^\otimes,$$

where the right-hand side is evidently an object in $\mathfrak{Sch}_{/S_0}^\otimes$. Thus, Δ_Z^\otimes is representable. Moreover, this representability implies (since both s^\otimes and t^\otimes are superétale) immediately that the projection $\pi_{R^\otimes \rightrightarrows U^\otimes}$ is representable, surjective, and superétale. Finally, by means of the isomorphism (31), the latter condition of (c-2) implies that the diagonal morphism $Z_t \rightarrow Z_t \times_{S_0} Z_t$ is separated and quasi-compact. This completes the proof of the implication (c) \Rightarrow (b), and consequently, the proof of Proposition 1.8.3. \square

2. LOGARITHMIC STRUCTURES ON SUPERSCHEMES

In this section, we shall give briefly a general formulation of log superschemes (or more generally, log superstacks). The notion of logarithmic structure on a superscheme, as well as a superstack (cf. Definition 2.1.1 (i)) is a supersymmetric generalization of the classical notion of logarithmic structure in the sense of J. M. Fontaine and L. Illusie. (Basic references for the notion of *logarithmic structure* on a scheme are, e.g., [13] and [11].) One of the most important concepts in log supergeometry is log supersmoothness (cf. Definition 2.4.1 (ii)). At the end of this section, we show (cf. Proposition 2.4.3 (ii) and Corollary 2.5.3 (i)-(iii)) how log supersmooth deformations of a log superstack or a morphism of log superstacks are controlled by the sheaf of logarithmic superderivations (cf. Definition 2.3.1 for the definition of a logarithmic superderivation).

2.1. Logarithmic structures.

Definition 2.1.1.

- (i) Let $X^\otimes := (X_b, \mathcal{O}_{X^\otimes})$ be a superscheme (resp., a superstack). A **logarithmic structure** (or **log structure** for short) on X^\otimes is a logarithmic structure $\alpha_{X_b} : \mathcal{M}_{X_b} \rightarrow \mathcal{O}_{X_b}$ on X_b (where \mathcal{M}_{X_b} denotes an étale sheaf of commutative monoids on X_b).

A **log superscheme** (resp., **log superstack**) is a triple

$$(35) \quad Y^{\otimes \log} := (Y_b, \mathcal{O}_{Y^\otimes}, \mathcal{M}_{Y_b} \xrightarrow{\alpha_{Y_b}} \mathcal{O}_{Y_b})$$

consisting of a superscheme (resp., a superstack) $Y^\otimes := (Y_b, \mathcal{O}_{Y^\otimes})$ and a log structure α_{Y_b} on Y_b (hence, of $Y^{\otimes \log}$). We shall refer to $Y_b^{\log} := (Y_b, \alpha_{Y_b})$ as the **underlying log scheme** (resp., **underlying log stack**) of $Y^{\otimes \log}$ and refer to Y^\otimes as the **underlying superscheme** (resp., **underlying superstack**) of $Y^{\otimes \log}$. Denote by $\beta_Y^{\otimes \log} : Y^{\otimes \log} \rightarrow Y_b^{\log}$ the morphism of log superschemes extending β_Y^\otimes .

- (ii) Let $X^{\otimes \log} := (X_b, \mathcal{O}_{X^\otimes}, \alpha_{X_b})$ and $Y^{\otimes \log} := (Y_b, \mathcal{O}_{Y^\otimes}, \alpha_{Y_b})$ be two log superschemes (resp., log superstacks). A **morphism of log superschemes** (resp., **morphism of log superstacks**) from $Y^{\otimes \log}$ to $X^{\otimes \log}$ is a triple

$$(36) \quad f^{\otimes \log} := (f_b : Y_b \rightarrow X_b, f^b : f_b^*(\mathcal{O}_{X^\otimes}) \rightarrow \mathcal{O}_{Y^\otimes}, f_b^\sharp : f_b^{-1}(\mathcal{M}_{X_b}) \rightarrow \mathcal{M}_{Y_b}),$$

where $f^\otimes := (f_b, f^b)$ forms a morphism $Y^\otimes \rightarrow X^\otimes$ between the underlying superschemes (resp., underlying superstacks) and $f_b^{\log} := (f_b, f_b^\sharp)$ forms a morphism $Y_b^{\log} \rightarrow X_b^{\log}$ between the underlying log schemes (resp., underlying log stacks).

Definition 2.1.2.

An **fs log superscheme** (resp., **fs log superstack**) is a log superscheme (resp., log superstack) whose underlying log scheme (resp., underlying log stack) is fine and saturated.

Let $\underline{X}^{\log} := (\underline{X}, \alpha_{\underline{X}})$ be an fs log scheme over S_0 and \mathcal{E} a coherent $\mathcal{O}_{\underline{X}}$ -module. Then, we shall write

$$(37) \quad \langle \underline{X}, \mathcal{E} \rangle^{\otimes \log}$$

for the log superscheme defined to be $\langle \underline{X}, \mathcal{E} \rangle$ (cf. (13)) equipped with the log structure pulled-back from \underline{X}^{\log} via $\langle \beta \rangle_{\underline{X}, \mathcal{E}}^{\otimes}$ (cf. (14)).

Definition 2.1.3.

Let $X^{\otimes \log}$ and $Y^{\otimes \log}$ be log superschemes (resp., log superstacks) and $f^{\otimes \log} : Y^{\otimes \log} \rightarrow X^{\otimes \log}$ a morphism of log superschemes (resp., a morphism of log

superstacks). We shall say that $f^{\otimes \log}$ is **strict superétale** (resp., a **strict closed immersion**) if f^{\otimes} is superétale (resp., a closed immersion) and f_b^{\log} is strict, in the sense of [11], § 1.2.

We shall write

$$(38) \quad \mathfrak{Sch}_{/S_0}^{\otimes \log}$$

for the category whose *objects* are fs log superschemes and whose *morphisms* are morphisms of log superschemes. Also, write $\mathfrak{Sch}_{/S_0}^{\log}$ for the full subcategory of $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ consisting of fs log schemes (i.e., fs log superschemes $X^{\otimes \log}$ with $\mathcal{O}_{X_f} = 0$). The fiber products and finite coproducts in $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ exist, and $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ admits the Grothendieck pretopology given by strict superétale morphisms; we shall refer to it as the **strict superétale pretopology**. In a natural manner, any log superstack may be thought of as a stack over $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ with respect to the strict superétale pretopology.

2.2. Logarithmic superdifferentials.

Let $S^{\otimes \log} := (S_b, \mathcal{O}_{S^{\otimes}}, \alpha_{S_b})$ and $X^{\otimes \log} := (X_b, \mathcal{O}_{X^{\otimes}}, \alpha_{X_b})$ be fs log superschemes and $f^{\otimes \log} := (f_b, f^b, f_b^\sharp) : X^{\otimes \log} \rightarrow S^{\otimes \log}$ a morphism of log superschemes. In the following, we shall define (in a functorial manner) a “log super” analogue of the sheaf of relative differential 1-forms, i.e., an $\mathcal{O}_{X^{\otimes}}$ -supermodule $\Omega_{X^{\otimes \log}/S^{\otimes \log}}$ together with the universal derivation $d : \mathcal{O}_{X^{\otimes}} \rightarrow \Omega_{X^{\otimes \log}/S^{\otimes \log}}$ defined as follows. Let us write $\Delta_X^{\otimes} : X^{\otimes} \rightarrow X^{\otimes} \times_{S^{\otimes}} X^{\otimes}$ for the diagonal morphism and write $\mathcal{J} := \text{Ker}(\mathcal{O}_{X^{\otimes} \times_{S^{\otimes}} X^{\otimes}} \rightarrow \Delta_{X*}^{\otimes}(\mathcal{O}_{X^{\otimes}}))$. Then, we shall define

$$(39) \quad \Omega_{X^{\otimes}/S^{\otimes}} := \Delta_X^{\otimes*}(\mathcal{J}/\mathcal{J}^2)$$

and write $d : \mathcal{O}_{X^{\otimes}} \rightarrow \Omega_{X^{\otimes}/S^{\otimes}}$ for the $f_b^{-1}(\mathcal{O}_{S^{\otimes}})$ -linear morphism given by assigning $a \mapsto d(a) := \overline{(a \otimes 1 - 1 \otimes a)}$ for any local section $a \in \mathcal{O}_{X^{\otimes}}$. For example, if $X^{\otimes} = \mathbb{A}_{S^{\otimes}}^{m|n}$, then we have

$$(40) \quad \Omega_{X^{\otimes}/S^{\otimes}} \cong \left(\bigoplus_{i=1}^m \mathcal{O}_{X^{\otimes}} d(t_i) \right) \oplus \left(\bigoplus_{i=1}^n \mathcal{O}_{X^{\otimes}} d(\psi_i) \right),$$

where $d(t_i)$ ($i = 1, \dots, m$) are bosonic elements in $\Omega_{X^{\otimes}/S^{\otimes}}$ and $d(\psi_i)$ ($i = 1, \dots, n$) are fermionic elements.

Moreover, let us define the $\mathcal{O}_{X^{\otimes}}$ -supermodule $\Omega_{X^{\otimes \log}/S^{\otimes \log}}$ to be

$$(41) \quad \Omega_{X^{\otimes \log}/S^{\otimes \log}} := (\Omega_{X^{\otimes}/S^{\otimes}} \oplus (\mathcal{O}_{X^{\otimes}} \otimes_{\mathbb{Z}} \mathcal{M}_{X_b}^{\text{gr}})) / \mathcal{N},$$

where

- (i) $\mathcal{M}_{X_b}^{\text{gr}}$ denotes the groupification of \mathcal{M}_{X_b} whose local sections are bosonic (hence we obtain the \mathcal{O}_{X^\otimes} -supermodule $\mathcal{O}_{X^\otimes} \otimes_{\mathbb{Z}} \mathcal{M}_{X_b}^{\text{gr}}$);
- (ii) \mathcal{N} denotes the \mathcal{O}_{X^\otimes} -subsupermodule generated locally by local sections of the following forms:
 - $(d(\alpha_{X_b}(a)), 0) - (0, \alpha_{X_b}(a) \otimes a)$ with $a \in \mathcal{M}_{X_b}$;
 - $(0, 1 \otimes a)$ with $a \in \text{Im}(f_b^{-1}(\mathcal{M}_{S_b}) \xrightarrow{f_b^\#} \mathcal{M}_{X_b})$.

The class of $(0, 1 \otimes a)$ for $a \in \mathcal{M}_{X_b}$ in $\Omega_{X^\otimes \log / S^\otimes \log}$ is denoted by $d\log(a)$. Finally, we write

$$(42) \quad \mathcal{T}_{X^\otimes \log / S^\otimes \log} := \Omega_{X^\otimes \log / S^\otimes \log}^\vee.$$

i.e., the dual \mathcal{O}_{X^\otimes} -supermodule of $\Omega_{X^\otimes \log / S^\otimes \log}$.

The following Propositions 2.2.1 and 2.2.2 may be verified immediately.

Proposition 2.2.1.

Let us consider a cartesian square diagram

$$(43) \quad \begin{array}{ccc} Y^{\otimes \log} & \xrightarrow{f^{\otimes \log}} & X^{\otimes \log} \\ \downarrow & \square & \downarrow \\ T^{\otimes \log} & \longrightarrow & S^{\otimes \log} \end{array}$$

in $\mathfrak{Sch}_{/S_0}^{\otimes \log}$. Then, the natural \mathcal{O}_{Y^\otimes} -linear morphism

$$(44) \quad f^{\otimes*}(\Omega_{X^\otimes \log / S^\otimes \log}) \rightarrow \Omega_{Y^\otimes \log / T^\otimes \log}$$

is an isomorphism.

Proposition 2.2.2.

- (i) *Let $X^{\otimes \log}$ and Y^\otimes be fs log superschemes and $f^{\otimes \log} : Y^\otimes \rightarrow X^{\otimes \log}$ a morphism of log superschemes. Then, there exists an exact sequence*

$$(45) \quad f^{\otimes*}(\Omega_{X^\otimes \log / S^\otimes \log}) \rightarrow \Omega_{Y^\otimes \log / S^\otimes \log} \rightarrow \Omega_{Y^\otimes \log / X^\otimes \log} \rightarrow 0$$

of \mathcal{O}_{Y^\otimes} -supermodule.

- (ii) *Suppose further that $f^{\otimes \log}$ is strict superétale. Then, $\Omega_{Y^\otimes \log / X^\otimes \log} = 0$ and the first arrow $f^{\otimes*}(\Omega_{X^\otimes \log / S^\otimes \log}) \rightarrow \Omega_{Y^\otimes \log / S^\otimes \log}$ in (45) is an isomorphism.*

2.3. Logarithmic superderivations.

Let $S^{\otimes \log}$, $X^{\otimes \log}$, and $f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}$ be as at the beginning of the previous subsection.

Definition 2.3.1.

Let \mathcal{E} be an \mathcal{O}_{X^\otimes} -supermodule. A **logarithmic superderivation of $(\mathcal{O}_{X^\otimes}, \mathcal{M}_{X_b})$ (over $S^{\otimes \log}$) with value in \mathcal{E}** is a pair $\partial := (D, \delta)$, where

- D is a superderivation $\mathcal{O}_{X^\otimes} \rightarrow \mathcal{E}$ over S^{\otimes} , i.e., an $f_b^{-1}(\mathcal{O}_{S^\otimes})$ -linear morphism satisfying that

$$(46) \quad D(a \cdot b) = D(a) \cdot b + (-1)^{|D| \cdot |a|} a \cdot D(b)$$

for any local sections $a, b \in \mathcal{O}_X$ (where $|D|$ denotes the parity of D);

- δ is a monoid homomorphism $\mathcal{M}_{X_b} \rightarrow \mathcal{E}$ such that

$$(47) \quad D(\alpha_{X_b}(m)) = \alpha_{X_b}(m) \cdot \delta(m)$$

for any local section $m \in \mathcal{M}_{X_b}$;

- $D(f_b^{-1}(b)) = \delta(f_b^\sharp(n)) = 0$ for any sections $b \in \mathcal{O}_{S^\otimes}$ and $n \in \mathcal{M}_{S_b}$.

If $\partial := (D, \delta)$ is a logarithmic superderivation, then we usually just write $\partial(a)$ and $\partial(m)$ (where $a \in \mathcal{O}_{X^\otimes}$ and $m \in \mathcal{M}_{X_b}$) for $D(a)$ and $\delta(m)$ respectively.

Remark 2.3.2.

Let \mathcal{E} be an \mathcal{O}_{X^\otimes} -supermodule. Denote by

$$(48) \quad \text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E})$$

the set of logarithmic superderivations of $(\mathcal{O}_{X^\otimes}, \mathcal{M}_{X_b})$ over $S^{\otimes \log}$ with value in \mathcal{E} . The structure of \mathcal{O}_{X^\otimes} -supermodule on \mathcal{E} gives rise to a structure of $\Gamma(X_b, \mathcal{O}_{X^\otimes})$ -supermodule on $\text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E})$. In particular, $\text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E})$ decomposes as

$$(49) \quad \text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E}) = \text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E})_b \oplus \text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E})_f.$$

It is clear that there exists a universal logarithmic superderivation

$$(50) \quad d \in \text{Def}_{S^\otimes}(X^{\otimes \log}; \Omega_{X^{\otimes \log}/S^{\otimes \log}})_b.$$

That is to say, the morphism

$$(51) \quad \begin{array}{ccc} \text{Hom}_{\mathcal{O}_{X^\otimes}}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}, \mathcal{E}) & \xrightarrow{\sim} & \text{Def}_{S^\otimes}(X^{\otimes \log}; \mathcal{E}) \\ h & \mapsto & h \circ d \end{array}$$

is an isomorphism of $\Gamma(X_b, \mathcal{O}_{X^\otimes})$ -supermodules. In particular, (since the isomorphism (51) is compatible with restriction to each open subscheme of X_b) the case of $\mathcal{E} = \mathcal{O}_{X^\otimes}$ implies that the dual $\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}$ of $\Omega_{X^{\otimes \log}/S^{\otimes \log}}$ is isomorphic to the sheaf given by assigning $U \mapsto \text{Def}_{S^\otimes}(X^{\otimes \log} \times_{X_b} U; \mathcal{E}|_U)$ (for any open subscheme U of X_b). By taking account of [18], Proposition 1.1.7, one verifies that $\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}$ admits a structure of Lie superalgebra over $f_b^{-1}(\mathcal{O}_{S^\otimes})$ with bracket operation given by

$$(52) \quad [\partial_1, \partial_2] := (D_1 \circ D_2 - (-1)^{|D_1| \cdot |D_2|} D_2 \circ D_1, D_1 \circ \delta_2 - (-1)^{|D_1| \cdot |D_2|} D_2 \circ \delta_1)$$

for any homogenous logarithmic superderivations $\partial_1 := (D_1, \delta_1)$ and $\partial_2 := (D_2, \delta_2)$.

Remark 2.3.3.

The discussions in §2.2 and §2.3 (especially, Propositions 2.2.1 and 2.2.2) generalize naturally to the case where $X^{\otimes \log}$ is a log superstack. In fact, $\Omega_{X^{\otimes \log}/S^{\otimes \log}}$ is constructed in such a way that if $t^{\otimes \log} : T^{\otimes \log} \rightarrow X^{\otimes \log}$ (where $T^{\otimes \log}$ is a superscheme) is a strict superétale morphism, then we have a functorial (with respect to $T^{\otimes \log}$) isomorphism $t^{\otimes *}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}) \cong \Omega_{T^{\otimes \log}/S^{\otimes \log}}$.

2.4. Log supersmooth morphisms.

Let $S^{\otimes \log}$ be an fs log superscheme, $X^{\otimes \log}$ an fs log superstack, and $f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}$ a morphism of log superstacks.

Definition 2.4.1.

Let m, n be nonnegative integers.

(i) An $(m|n)$ -**chart** on $X^{\otimes \log}/S^{\otimes \log}$ is a triple

$$(53) \quad (Y^{\otimes \log}, U^{\log}, \eta^{\otimes \log}),$$

where

- $Y^{\otimes \log}$ is an affine fs log superscheme together with a strict superétale morphism $Y^{\otimes \log} \rightarrow X^{\otimes \log}$ over S_0 ;
- U^{\log} is an fs log affine scheme together with an integral log smooth morphism $U^{\log} \rightarrow S_b^{\log}$ of relative dimension m ;
- $\eta^{\otimes \log}$ is an isomorphism $Y^{\otimes \log} \xrightarrow{\sim} U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes \log}}^{0|n}$ over $S^{\otimes \log}$.

(ii) We shall say that $X^{\otimes \log}$ is **log supersmooth over $S^{\otimes \log}$ of relative superdimension $m|n$** if there exists a collection $\{(Y_\gamma^{\otimes \log}, U_\gamma^{\log}, \eta_\gamma^{\otimes \log})\}_\gamma$ of $(m|n)$ -charts on $X^{\otimes \log}/S^{\otimes \log}$ for which the morphism $\coprod_\gamma Y_\gamma^{\otimes \log} \rightarrow X^{\otimes \log}$ is a strict superétale covering of $X^{\otimes \log}$.

Remark 2.4.2.

It is clear that if both $S^{\otimes \log}$ and $X^{\otimes \log}$ has trivial structures, then $X^{\otimes \log}$ is log supersmooth over $S^{\otimes \log}$ of relative superdimension $m|n$ if and only if $X^{\otimes \log}$ is supersmooth of relative superdimension $m|n$, in the sense of Definition 1.7.2, (ii).

Proposition 2.4.3.

Suppose that $X^{\otimes \log}$ is log supersmooth over $S^{\otimes \log}$ of relative superdimension $m|n$ for some nonnegative integers m, n . Then, the following assertions hold.

- (i) The $\mathcal{O}_{X^{\otimes \log}/S^{\otimes \log}}$ -supermodule $\Omega_{X^{\otimes \log}/S^{\otimes \log}}$ is a supervector bundle of superrank $m|n$.
- (ii) Let us consider a commutative square diagram

$$(54) \quad \begin{array}{ccc} T^{\otimes \log} & \xrightarrow{t_X^{\otimes \log}} & X^{\otimes \log} \\ i^{\otimes \log} \downarrow & & \downarrow f^{\otimes \log} \\ \tilde{T}^{\otimes \log} & \xrightarrow{\tilde{t}_S^{\otimes \log}} & S^{\otimes \log}, \end{array}$$

where $T^{\otimes \log}$ is affine and $i^{\otimes \log}$ is a strict closed immersion defined by a square nilpotent superideal \mathcal{J} of $\mathcal{O}_{\tilde{T}^{\otimes \log}}$. (Hence, \mathcal{J} may be thought of as an $\mathcal{O}_{T^{\otimes \log}}$ -supermodule.) We shall write

$$(55) \quad \mathcal{F} := \mathcal{H}om_{\mathcal{O}_{T^{\otimes \log}}}(t_X^{\otimes *}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}), \mathcal{J})_b \left(\cong (t_X^{\otimes *}(\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}) \otimes \mathcal{J})_b \right)$$

i.e., the $\mathcal{O}_{T^{\otimes \log}}$ -submodule of $\mathcal{H}om_{\mathcal{O}_{T^{\otimes \log}}}(t_X^{\otimes *}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}), \mathcal{J})$ consisting of $\mathcal{O}_{T^{\otimes \log}}$ -linear homomorphisms $t_X^{\otimes *}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}) \rightarrow \mathcal{J}$ of even parity. Also, we shall write

$$(56) \quad \mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})$$

for the strict superétale sheaf on $\tilde{T}^{\otimes \log}$ which, to any strict superétale morphism $\alpha^{\otimes \log} : \tilde{T}_1^{\otimes \log} \rightarrow \tilde{T}^{\otimes \log}$, assigns the set of morphisms $\tilde{t}_{1,X}^{\otimes \log} : \tilde{T}_1^{\otimes \log} \rightarrow X^{\otimes \log}$ which makes the diagram

$$(57) \quad \begin{array}{ccc} T^{\otimes \log} \times_{\tilde{T}^{\otimes \log}} \tilde{T}_1^{\otimes \log} & \longrightarrow & X^{\otimes \log} \\ i^{\otimes \log} \downarrow & \nearrow \tilde{t}_{1,X}^{\otimes \log} & \downarrow f^{\otimes \log} \\ \tilde{T}_1^{\otimes \log} & \xrightarrow{\tilde{t}_S^{\otimes \log} \circ \alpha^{\otimes \log}} & S^{\otimes \log}, \end{array}$$

commute, where the upper horizontal arrow denotes the composite of $t_X^{\otimes \log}$ and the projection $T^{\otimes \log} \times_{\tilde{T}^{\otimes \log}} \tilde{T}_1^{\otimes \log} \rightarrow T^{\otimes \log}$ to the first factor.

Then, $\mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})$ is nonempty (i.e., admits locally a section), and moreover, admits canonically a structure of affine space

$$(58) \quad \begin{aligned} \mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log}) \times i_{b*}(\mathcal{F}) &\rightarrow \mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log}) \\ (\tilde{t}_X^{\otimes \log}, \partial) &\mapsto \tilde{t}_X^{\otimes \log} \boxplus^\dagger \partial \end{aligned}$$

modeled on $i_{b*}(\mathcal{F})$.

Proof. Assertion (i) follows from (40), Proposition 2.2.2 (ii), and [13], Proposition (3.10). Next, we shall prove the former assertion of (ii), i.e., that $\mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})$ is nonempty. After possibly replacing $\tilde{T}^{\otimes \log}$ with its strict superétale covering, one may assume, without loss of generality, that $X^{\otimes \log} = U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|n}$ for some fs log affine scheme U^{\log} together with an integral log smooth morphism $f_U^{\log} : U^{\log} \rightarrow S_b^{\log}$ of relative dimension m . Consider the commutative diagram

$$(59) \quad \begin{array}{ccc} T_b^{\log} & \xrightarrow{\underline{t}_X^{\log}} & U^{\log} \\ i_b^{\log} \downarrow & & \downarrow f_U^{\log} \\ \tilde{T}_b^{\log} & \xrightarrow{(\tilde{t}_S)_b^{\log}} & S_b^{\log}, \end{array}$$

where the upper horizontal arrow \underline{t}_X^{\log} denotes the composite $(t_X)_b^{\log} : T_b^{\log} \rightarrow X_b^{\log}$ and the natural projection $X_b^{\log} \rightarrow U^{\log}$. Since f_U^{\log} is log smooth, there exists a morphism $\tilde{t}_{b,U}^{\log} : \tilde{T}_b^{\log} \rightarrow U^{\log}$ such that $\tilde{t}_{b,U}^{\log} \circ i_b^{\log} = \underline{t}_X^{\log}$ and $f_U^{\log} \circ \tilde{t}_{b,U}^{\log} = (\tilde{t}_S)_b^{\log}$. On the other hand, let us consider the functorial bijection (17) obtained in Proposition 1.4.3 (of the case where $(m, n) = (0, 1)$). Then, the composite of $t_X^{\otimes \log} : T^{\otimes \log} \rightarrow U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1}$ and the projection $U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1} \rightarrow \mathbb{A}_{S^{\otimes}}^{0|1}$ to the second factor extends, strict superétale locally on $\tilde{T}^{\otimes \log}$, to a morphism $\tilde{t}_{\mathbb{A}}^{\otimes \log} : \tilde{T}^{\otimes \log} \rightarrow \mathbb{A}_{S^{\otimes}}^{0|1}$. Thus, the morphism

$$(60) \quad (\tilde{t}_{b,U}^{\log} \circ \beta_{\tilde{T}}^{\otimes \log}, \tilde{t}_{\mathbb{A}}^{\otimes \log}) : \tilde{T}^{\otimes \log} \rightarrow U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1}$$

determines a section of $\mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})$.

Next, we shall prove the latter assertion of (ii). One may assume, without loss of generality, that there exists an element $\tilde{t}_{X,0}^{\otimes \log} \in \Gamma(\tilde{T}_b, \mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log}))$. Since we have an isomorphism

$$(61) \quad \Gamma(\tilde{T}_b, i_{b*}(\mathcal{F})) (\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X^{\otimes}}}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}, t_{X*}^{\otimes}(\mathcal{J}))_b) \xrightarrow{\sim} \mathrm{Der}_{S^{\otimes \log}}(X^{\otimes \log}; t_{X*}^{\otimes}(\mathcal{J}))_b$$

(cf. 51), it suffices to construct a functorial (with respect to $\tilde{T}^{\otimes \log}$) bijection

$$(62) \quad \mathrm{Der}_{S^{\otimes \log}}(X^{\otimes \log}; (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{J}))_b \xrightarrow{\sim} \Gamma(\tilde{T}_b, \mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})).$$

Denote by $\tilde{t}_{X,0}^{\otimes \dagger} : \mathcal{O}_{X^{\otimes}} \rightarrow t_{X*}^{\otimes}(\mathcal{O}_{\tilde{T}^{\otimes}})$ and $\tilde{t}_{X,0}^{\otimes \dagger} : \mathcal{M}_{X_b} \rightarrow t_{X*}^{\otimes}(\mathcal{M}_{\tilde{T}_b})$ the morphism arising naturally from $\tilde{t}_{X,0}^{\otimes}$ (where we consider both $\mathcal{O}_{\tilde{T}^{\otimes}}$ and $\mathcal{M}_{\tilde{T}_b}$ as sheaves on T_b via the underlying homeomorphism between topological spaces of $i^{\otimes \log}$). Let us take an element

$$(63) \quad \partial := (\mathcal{O}_{X^{\otimes}} \xrightarrow{D} (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{J}), \mathcal{M}_{X_b} \xrightarrow{\delta} (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{J}))$$

of $\text{Der}_{S^{\otimes \log}}(X^{\otimes \log}; (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{J}))_b$. By applying the inclusions

$$(64) \quad (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{J}) \hookrightarrow (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{O}_{\tilde{T}^{\otimes}}) \quad \text{and} \quad (\tilde{t}_{X,0}^{\otimes})_*(1 + \mathcal{J}) \hookrightarrow (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{M}_{\tilde{T}_b}),$$

one may obtain two maps $\tilde{t}_{X,0}^{\otimes \dagger} + D$ and $\tilde{t}_{X,0}^{\otimes \dagger} + \delta$ given by

$$(65) \quad \begin{aligned} \tilde{t}_{X,0}^{\otimes \dagger} + D : \mathcal{O}_{X^{\otimes}} &\rightarrow (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{O}_{\tilde{T}^{\otimes}}), & \tilde{t}_{X,0}^{\otimes \dagger} + \delta : \mathcal{M}_{X_b} &\rightarrow (\tilde{t}_{X,0}^{\otimes})_*(\mathcal{M}_{\tilde{T}_b}) \\ a &\mapsto a + D(a) & b &\mapsto 1 + \delta(b) \end{aligned}$$

for any local sections $a \in \mathcal{O}_{X^{\otimes}}$, $b \in \mathcal{M}_{X_b}$. By the definition of a logarithmic superderivation, the pair $(\tilde{t}_{X,0}^{\otimes \dagger} + D, \tilde{t}_{X,0}^{\otimes \dagger} + \delta)$ determines a new morphism $\tilde{t}_{X,0}^{\otimes} \boxplus^{\dagger} \partial : \tilde{T}^{\otimes \log} \rightarrow X^{\otimes \log}$ in $\mathcal{D}ef_{\tilde{T}^{\otimes \log}}(t_X^{\otimes \log})$. One verifies immediately that this assignment $\partial \mapsto \tilde{t}_{X,0}^{\otimes} \boxplus^{\dagger} \partial$ determines the desired bijection (62). This completes the proof of Proposition 2.4.3. \square

2.5. Log supersmooth liftings.

Definition 2.5.1.

Let $X^{\otimes \log}$ and $S^{\otimes \log}$ be as in Proposition 2.4.3. Also, let $i_S^{\otimes \log} : S^{\otimes \log} \rightarrow \tilde{S}^{\otimes \log}$ be a strict closed immersion determined by a nilpotent superideal \mathcal{J} on $\mathcal{O}_{\tilde{S}^{\otimes}}$.

(i) By a **log supersmooth lifting** of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$, we mean a triple

$$(66) \quad \tilde{\mathbb{X}} := (\tilde{X}^{\otimes \log}, \tilde{f}^{\otimes \log}, i_X^{\otimes \log})$$

consisting of a log superstack $\tilde{X}^{\otimes \log}$, a log supersmooth morphism $\tilde{f}^{\otimes \log} : \tilde{X}^{\otimes \log} \rightarrow \tilde{S}^{\otimes \log}$, and a strict closed immersion $i_X^{\otimes \log} : X^{\otimes \log} \rightarrow \tilde{X}^{\otimes \log}$ which make the square diagram

$$(67) \quad \begin{array}{ccc} X^{\otimes \log} & \xrightarrow{i_X^{\otimes \log}} & \tilde{X}^{\otimes \log} \\ f^{\otimes \log} \downarrow & & \downarrow \tilde{f}^{\otimes \log} \\ S^{\otimes \log} & \xrightarrow{i_S^{\otimes \log}} & \tilde{S}^{\otimes \log} \end{array}$$

commute and cartesian.

(ii) Let $\tilde{\mathbb{X}}_l := (\tilde{X}_l^{\otimes \log}, \tilde{f}_l^{\otimes \log}, i_{X_l}^{\otimes \log})$ ($l = 1, 2$) be log supersmooth liftings of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$. An **isomorphism of log supersmooth liftings** from $\tilde{\mathbb{X}}_1$ to $\tilde{\mathbb{X}}_2$ is an isomorphism $j^{\otimes \log} : \tilde{X}_1^{\otimes \log} \xrightarrow{\sim} \tilde{X}_2^{\otimes \log}$ such that $\tilde{f}_2^{\otimes \log} \circ j^{\otimes \log} = \tilde{f}_1^{\otimes \log}$ and $i_{X_2}^{\otimes \log} \circ j^{\otimes \log} = i_{X_1}^{\otimes \log}$.

Remark 2.5.2.

Suppose that we are given a log supersmooth lifting $\tilde{\mathbb{X}} := (\tilde{X}^{\otimes \log}, \tilde{f}^{\otimes \log}, i_X^{\otimes \log})$ of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$ and a strict superétale morphism $Y^{\otimes \log} \rightarrow X^{\otimes \log}$ over $S^{\otimes \log}$.

Then, by Proposition 1.2.5, there exists a log supersmooth lifting of $Y^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$ which is uniquely determined up to isomorphism. We denote this log supersmooth lifting by

$$(68) \quad \tilde{X}|_{Y^{\otimes \log}} := (\tilde{X}^{\otimes \log}|_{Y^{\otimes \log}}, \tilde{f}^{\otimes \log}|_{Y^{\otimes \log}}, i_X^{\otimes \log}|_{Y^{\otimes \log}}).$$

Corollary 2.5.3.

Let us keep the notation in Definition 2.5.1. Suppose further that \mathcal{J} is square nilpotent (hence \mathcal{J} may be thought of as an $\mathcal{O}_{S^{\otimes}}$ -supermodule.) Also, write

$$(69) \quad \mathcal{F} := \mathcal{H}om_{\mathcal{O}_{X^{\otimes}}}(\Omega_{X^{\otimes \log}/S^{\otimes \log}}, \mathcal{J}\mathcal{O}_{X^{\otimes}})_b.$$

(Note that, since X^{\otimes} is superflat over S^{\otimes} , we have $\mathcal{F} \cong (\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}} \otimes f^{\otimes *}(\mathcal{J}))_b$.)

- (i) Suppose that we are given a log supersmooth lifting $\tilde{X} := (\tilde{X}^{\otimes \log}, \tilde{f}^{\otimes \log}, i_X^{\otimes \log})$ of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$. Then, the group of automorphisms of \tilde{X} is canonically isomorphic to $\Gamma(X_b, \mathcal{F})$.
- (ii) Suppose that we are given two log smooth liftings $\tilde{X}_l := (\tilde{X}_l^{\otimes \log}, \tilde{f}_l^{\otimes \log}, i_{\tilde{X}_l}^{\otimes \log})$ ($l = 1, 2$) of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$. Then, there exists a strict superétale covering $Y^{\otimes \log} \rightarrow X^{\otimes \log}$ such that $\tilde{X}_1|_{Y^{\otimes \log}} \xrightarrow{\sim} \tilde{X}_2|_{Y^{\otimes \log}}$. In particular, if there exists a log supersmooth lifting of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$, then the set of isomorphism classes of log supersmooth liftings of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$ forms canonically an affine space modeled on $H^1(X_b, \mathcal{F})$.
- (iii) A log supersmooth lifting of $X^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$ exists if $H^2(X_b, \mathcal{F}) = 0$.

Proof. Assertions (i), (ii), and (iii) follow from Proposition 2.4.3, (ii) together with a routine argument in the theory of the classical (log) smoothness. \square

3. STABLE LOG TWISTED SUSY₁ CURVES

In this section, we shall consider, by means of various notions defined previously, supersymmetric analogues of a pointed log twisted curve (with a canonical logarithmic structure). We first recall (in § 3.1) the definition of a twisted curve and prove the Riemann-Roch theorem for twisted curves (cf. Theorem 3.1.2), which will be used in, e.g., computing the superdimension of the relevant moduli introduced later. Then, log twisted (1|1)-curves (cf. Definition 3.2.3 (i)) are defined and characterized by local models, which are the fiber products of a (locally defined) log twisted curve and the affine superspace of superdimension 0|1. Moreover, by introducing a logarithmic and twisted analogue of superconformal structure, we obtain the notion of a (pointed) log twisted SUSY₁ curve (cf. Definition 3.4.1) which are central objects of the present paper. As shown in Corollary 3.4.4, it is a basic property that suitable (with respect to

log supersmooth deformation) local models of a (pointed) log twisted SUSY₁ curve may be chosen. Finally, we introduce the fibered category ${}_{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ (cf. (119)) classifying stable log twisted SUSY₁ curve of prescribed type (g, r, λ) (cf. Definition 3.6.1).

3.1. The Riemann-Roch theorem for twisted curves.

In this section, let us review the notion of a twisted curve and consider the Riemann-Roch theorem for twisted curves. Here, recall that the *tameness* condition on a Deligne-Mumford stack Z means that for every geometric point $q : \text{Spec}(k) \rightarrow Z$ the group $\text{Aut}(q)$ of its stabilizers has order prime to the characteristic of the algebraically closed field k .)

Definition 3.1.1.

Let \underline{S} be a scheme.

- (i) A **local twisted curve** over \underline{S} is a flat morphism $\underline{f} : \underline{U} \rightarrow \underline{S}$ of tame Deligne-Mumford stacks satisfying the following three conditions (i-1)-(i-3):
- (i-1) The geometric fibers of \underline{f} are purely 1-dimensional and, étale locally on \underline{U} , isomorphic to nodal curves;
 - (i-2) The smooth locus $\underline{U}^{\text{sm}}$ of \underline{U} (over \underline{S}) is an algebraic space;
 - (i-3) For each node q of a geometric fiber of \underline{f} , there exists a commutative diagram:

$$(70) \quad \begin{array}{ccccc} V & \xrightarrow{c} & [T/\mu_{l'}] & \xrightarrow{b} & R \\ & \searrow d & & & \downarrow a \\ & & \underline{U} & \xrightarrow{\underline{f}} & \underline{S}, \end{array}$$

where

- $R = \text{Spec}(A)$ for some commutative ring A and a denotes an étale neighborhood of $\underline{f}(q) \in \underline{S}$;
 - $T = \text{Spec}(A[z, w]/(zw - t))$ for some $t \in A$;
 - $[T/\mu_{l'}]$ denotes the quotient stack of T by $\mu_{l'}$, where l' is a positive integer, $\mu_{l'}$ denotes the group scheme over R of l' -th roots of unity, and the action of $\mu_{l'}$ is given by $(z, w) \mapsto (\xi \cdot z, \xi^{-1} \cdot w)$ for any $\xi \in \mu_{l'}$.
 - b denotes the natural projection, c denotes an étale morphism and d is an étale neighborhood of q .
- (ii) Let g be a nonnegative integer. A **twisted curve (of genus g)** over \underline{S} is a local twisted curve over \underline{S} which is proper and whose coarse moduli space becomes a semistable curve (of genus g) over \underline{S} (cf. [5], Definition 2.4.1).

We prove the following assertion, which is the Riemann-Roch theorem for line bundles on a twisted curve.

Theorem 3.1.2.

Let X be a twisted curve of genus g (≥ 0) over an algebraically closed field k , and let \mathcal{L} be a line bundle on X of total degree m . (Here, it follows from [5], Proposition 2.5.6, that m is necessarily an integer.) We shall write

$$(71) \quad \chi(X, \mathcal{L}) := \dim_k(H^0(X, \mathcal{L})) - \dim_k(H^1(X, \mathcal{L})).$$

Then, we have that $H^2(X, \mathcal{L}) = 0$ and $\chi(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$.

Proof. Write $|X|$ for the coarse moduli space of X and $\pi : X \rightarrow |X|$ for the projection. First, we shall compare $\chi(X, \mathcal{L})$ with $\chi(|X|, \pi_*(\mathcal{L}))$ (i.e., the Euler characteristic of $\pi_*(\mathcal{L})$ in the classical sense). Denote by $e_1, \dots, e_s : \text{Spec}(k) \rightarrow |X|$ the set of nodes of $|X|$. For each $i \in \{1, \dots, s\}$, the fiber product $X \times_{|X|, e_i} \text{Spec}(k)$ is isomorphic to the classifying stack $B(\text{Aut}(\tilde{e}_i))$ of the group $\text{Aut}(\tilde{e}_i)$ of stabilizers of the (unique) point $\tilde{e}_i \in X(k)$ over e_i . Since $\text{Aut}(\tilde{e}_i)$ is, by the definition of a twisted curve, isomorphic to μ_{l_i} for some $l_i \geq 1$, the pull-back $\tilde{e}_i^*(\mathcal{L})$ of \mathcal{L} may be thought of as the trivial k -module k (of rank one) equipped with a μ_{l_i} -action. Write

$$(72) \quad E^{\text{nt}} := \{i \mid 1 \leq i \leq s \text{ and the } \mu_{l_i}\text{-action on } \tilde{e}_i^*(\mathcal{L}) \text{ is nontrivial}\}.$$

If $i \in E^{\text{nt}}$, then one verifies immediately that

$$(73) \quad H^q(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L})) = 0 \ (q = 0, 2) \text{ and } H^1(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L})) = k.$$

On the other hand, if $i \notin E^{\text{nt}}$, then

$$(74) \quad H^q(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L})) = 0 \ (q = 1, 2) \text{ and } H^0(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L})) = k.$$

Now, let us consider the Leray spectral sequence

$$(75) \quad E_2^{p,q} := H^p(|X|, \mathbb{R}^q \pi_*(\mathcal{L})) \implies H^{p+q}(X, \mathcal{L})$$

associated with \mathcal{L} . Since $\mathbb{R}^q \pi_*(-)$ ($q \geq 1$) vanishes on the smooth locus $|X|^{\text{sm}}$ of $|X|$, we have that

$$(76) \quad H^j(|X|, \mathbb{R}^i \pi_*(-)) = 0 \text{ unless } j = 0 \text{ or } (i, j) = (0, 1).$$

Moreover, $\mathbb{R}^q \pi_*(\mathcal{L})$ is isomorphic to $\bigoplus_{i=1}^s e_{i*}(H^q(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L})))$. Hence, it follows from (73), (74), and (76) that $H^2(X, \mathcal{L}) = 0$ (which completes the proof of the former equality) and

$$(77) \quad \begin{aligned} & \dim_k(H^1(X, \mathcal{L})) - \dim_k(H^1(|X|, \pi_*(\mathcal{L}))) \\ &= \sum_{i \in E^{\text{nt}}} \dim_k(H^1(\mu_{l_i}, \tilde{e}_i^*(\mathcal{L}))) \\ &= \sharp E^{\text{nt}}. \end{aligned}$$

Since the equality $H^0(X, \mathcal{L}) = H^0(|X|, \pi_*(\mathcal{L}))$ is evidently verified, we have

$$(78) \quad \chi(X, \mathcal{L}) - \chi(|X|, \pi_*(\mathcal{L})) = \sharp E^{\text{nt}}.$$

Next, we shall compare the total degree of \mathcal{L} and $\pi^*(\pi_*(\mathcal{L}))$. For each $i \in \{1, \dots, s\}$, the formal neighborhood \tilde{T}_i of X at \tilde{e}_i is isomorphic to the quotient stack $[\text{Spec}(R)/\mu_{l_i}]$, where $R := k[[z, w]]/(zw)$ and the μ_{l_i} -action on $\text{Spec}(R)$ is given by $(z, w) \mapsto (\xi \cdot z, \xi^{-1} \cdot w)$ for any $\xi \in \mu_{l_i}$. In particular, if T_i denotes the formal neighborhood of $|X|$ at e_i , then we have $T_i \cong \text{Spec}(R^{l_i})$, where $R^{l_i} := k[[z^{l_i}, w^{l_i}]]/(z^{l_i}y^{l_i})$, and the morphism $\tilde{T}_i \rightarrow T_i$ induced by π is given by the natural inclusion $R^{l_i} \hookrightarrow R$. A choice of trivialization $\mathcal{L}|_{\text{Spec}(R)} \xrightarrow{\sim} \mathcal{O}_{\text{Spec}(R)}$ allows us to identify the total space of the line bundle $\mathcal{L}|_{\tilde{T}_i}$ with the quotient stack $[(\tilde{T}_i \times \text{Spec}(k[t]))/\mu_{l_i}]$, where the μ_{l_i} -action is given by $(z, w, t) \mapsto (\xi \cdot z, \xi^{-1} \cdot w, \xi^{m_i} \cdot t)$ for some integer m_i with $0 \leq m_i \leq l$. Then, \mathcal{O}_{T_i} -module $\pi_*(\mathcal{L})|_{T_i}$ corresponds to the ideal $(z^{l-m_i}, w^{m_i}) \subseteq R^{l_i}$. The restriction to \tilde{T}_i of the natural morphism $\pi^*(\pi_*(\mathcal{L})) \rightarrow \mathcal{L}$ may be identified with the natural inclusion of the ideal $(z^{l-m_i}, w^{m_i}) \subseteq R$. If $i \notin E^{\text{nt}}$ (resp., $i \in E^{\text{nt}}$), then the length of $\mathcal{L}/\pi^*(\pi_*(\mathcal{L}))$ at \tilde{e}_i is 0 (resp., $\frac{1}{l} \cdot \text{length}(R/(z^{l-m_i}, w^{m_i})) = 1$). Since $\pi^*(\pi_*(\mathcal{L})) \rightarrow \mathcal{L}$ is injective and its cokernel is only supported at $\bigcup_{i=1}^s \text{Im}(\tilde{e}_i)$, we have

$$(79) \quad \deg(\mathcal{L}) = \deg(\pi^*(\pi_*(\mathcal{L}))) + \sum_{i \in E^{\text{nt}}} 1 = \deg(\pi_*(\mathcal{L})) + \sharp E^{\text{nt}}.$$

By combining (78) and (79), we have the equality $\chi(X, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$, as desired. \square

3.2. Log twisted (1|1)-curves.

Let \underline{S} be a scheme and $\underline{f} : \underline{U} \rightarrow \underline{S}$ a local twisted curve over \underline{S} . According to (the proof of) [19], Theorem 3.6, there exist canonically log structures

$$(80) \quad \alpha_{\underline{U}}^{\underline{f}} : \mathcal{M}_{\underline{U}} \rightarrow \mathcal{O}_{\underline{U}} \quad \text{and} \quad \alpha_{\underline{S}}^{\underline{f}} : \mathcal{M}_{\underline{S}} \rightarrow \mathcal{O}_{\underline{S}}$$

on \underline{U} and \underline{S} respectively (where we denote the resulting log stacks by $\underline{U}^{\underline{f}\text{-log}}$ and $\underline{S}^{\underline{f}\text{-log}}$ respectively), and moreover, a special morphism

$$(81) \quad \underline{f}^{\underline{f}\text{-log}} := (\underline{f}, \underline{f}^{\flat} : \underline{f}^{-1}(\mathcal{M}_{\underline{S}}) \rightarrow \mathcal{M}_{\underline{U}}) : \underline{U}^{\underline{f}\text{-log}} \rightarrow \underline{S}^{\underline{f}\text{-log}}$$

(cf. [19], Theorem 3.5 for the definition of “*special*”) extending \underline{f} . The data $(\alpha_{\underline{U}}^{\underline{f}}, \alpha_{\underline{S}}^{\underline{f}}, \underline{f}^{\underline{f}\text{-log}})$ is uniquely determined up to unique isomorphism.

Definition 3.2.1.

Let \underline{S} be a scheme and $\alpha_{\underline{S}}^1 : \mathcal{M}_{\underline{S}}^1 \rightarrow \mathcal{O}_{\underline{S}}$, $\alpha_{\underline{S}}^2 : \mathcal{M}_{\underline{S}}^2 \rightarrow \mathcal{O}_{\underline{S}}$ two log structures on \underline{S} . We shall say that a morphism $(\underline{S}, \alpha_{\underline{S}}^1) \rightarrow (\underline{S}, \alpha_{\underline{S}}^2)$ of log schemes is **log-like** (over \underline{S}) if its underlying endomorphism of \underline{S} coincides with the identity morphism.

Definition 3.2.2.

Let \underline{S}^{\log} be an fs log scheme. A **log local twisted curve over \underline{S}^{\log}** is a morphism $\underline{f}^{\log} : \underline{U}^{\log} \rightarrow \underline{S}^{\log}$ of log stacks satisfying the following two conditions:

- (i) The underlying morphism $\underline{f} : \underline{U} \rightarrow \underline{S}$ is a local twisted curve over \underline{S} ;
- (ii) There exist a log-like morphisms $\underline{S}^{\log} \rightarrow \underline{S}^{f\text{-log}}$ and $\underline{U}^{\log} \rightarrow \underline{U}^{f\text{-log}}$ over \underline{S} and \underline{U} respectively which make the square diagram

$$(82) \quad \begin{array}{ccc} \underline{U}^{\log} & \longrightarrow & \underline{U}^{f\text{-log}} \\ \underline{f}^{\log} \downarrow & & \downarrow \underline{f}^{f\text{-log}} \\ \underline{S}^{\log} & \longrightarrow & \underline{S}^{f\text{-log}} \end{array}$$

commute and cartesian.

Now, let us fix an fs log superscheme $S^{\otimes \log}$.

Definition 3.2.3.

- (i) A **log twisted (1|1)-curve** over $S^{\otimes \log}$ is a log superstack $X^{\otimes \log}$ over $S^{\otimes \log}$ such that X^{\otimes}/S^{\otimes} is proper and, for each geometric point q of X_b , there exists a (1|1)-chart $(Y^{\otimes \log}, U^{\log}, \eta^{\otimes \log})$ on $X^{\otimes \log}/S^{\otimes \log}$ around q such that U^{\log} is a log local twisted curve over S_b^{\log} . We shall refer to such a (1|1)-chart $(Y^{\otimes \log}, U^{\log}, \eta^{\otimes \log})$ as a **log twisted (1|1)-chart** (around q) on $X^{\otimes \log}/S^{\otimes \log}$.
- (ii) Let $X^{\otimes \log}/S^{\otimes \log}$ be a log twisted (1|1)-curve over $S^{\otimes \log}$. Then, the induced stack X_t is a twisted curve over S_t . We shall say that $X^{\otimes \log}/S^{\otimes \log}$ is **of genus g** if X_t/S_t is of genus g in the sense of [5], Definition 2.4.1.

3.3. Pointed log twisted (1|1)-curves.

In the rest of the present paper, let us fix a pair of nonnegative integers (g, r) satisfying that $2g - 2 + r > 0$.

Definition 3.3.1.

An **r -pointed log twisted (1|1)-curve of genus g** over $S^{\otimes \log}$ is a collection of data

$$(83) \quad \mathfrak{X}^{\otimes \star} := (X^{\otimes \log}/S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r),$$

where

- $X^{\otimes \log}$ denotes a log twisted (1|1)-curve of genus g over $S^{\otimes \log}$;
- $[\sigma_i^{\otimes}]$ (for each $i = 1, \dots, r$) denotes a closed subsuperscheme of X^{\otimes} over S^{\otimes} represented by a closed immersion $\sigma_i^{\otimes} : \mathbb{A}_{S^{\otimes}}^{01} \rightarrow X^{\otimes}$ over S^{\otimes} ,

satisfying the following conditions:

- (i) $\mathbb{A}_{S^\otimes}^{0|1} \times_{\sigma_i^\otimes, X^\otimes, \sigma_j^\otimes} \mathbb{A}_{S^\otimes}^{0|1} = \emptyset$ for any pair (i, j) with $i \neq j$;
- (ii) The smooth locus X_t^{sm} of X_t (over S_t) may be represented by a scheme over S_t and the image $\text{Im}((\sigma_i)_t)$ of each $(\sigma_i)_t$ lies in X_t^{sm} .

Let $\mathfrak{X}^{\otimes\star} := (X^{\otimes\log}/S^{\otimes\log}, \{[\sigma_i^\otimes]\}_{i=1}^r)$ be an r -pointed log twisted $(1|1)$ -curve of genus g over $S^{\otimes\log}$. Then, the collection of data

$$(84) \quad \mathfrak{X}_t^\star := (X_t/S_t, \{[(\sigma_i)_t]\}_{i=1}^r)$$

forms an r -pointed twisted curve of genus g over S_t (in the sense of [3], Definition 4.1.2); we shall refer to it as the **underlying (pointed) twisted curve** of $\mathfrak{X}^{\otimes\star}$.

Proposition 3.3.2.

Let $\mathfrak{X}^{\otimes\star} := (X^{\otimes\log}/S^{\otimes\log}, \{[\sigma_i^\otimes]\}_{i=1}^r)$ be an r -pointed twisted $(1|1)$ -curve of genus g over $S^{\otimes\log}$. (We shall fix a representative σ_i^\otimes of $[\sigma_i^\otimes]$ for each i .) Also, we shall fix $i \in \{1, \dots, r\}$ and a geometric point q of $\text{Im}([\sigma_i])$. Then, there exists a collection of data

$$(85) \quad \mathbb{U}^\star := (Y^{\otimes\log} \xrightarrow{\pi^{\otimes\log}} X^{\otimes\log}, U^{\log}, \eta^{\otimes\log}, \Sigma^U, \sigma^U),$$

where

- the triple $(Y^{\otimes\log}, U^{\log}, \eta^{\otimes\log})$ is a log twisted $(1|1)$ -chart around q of $X^{\otimes\log}/S^{\otimes\log}$;
- $\pi^{\otimes\log}$ denotes the structure morphism of $Y^{\otimes\log}$ over $X^{\otimes\log}$;
- Σ^U is an étale scheme over S_b ;
- σ^U is a closed immersion $\Sigma^U \rightarrow U$ over S_b .

such that the square diagram

$$(86) \quad \begin{array}{ccc} \Sigma^U \times_{S_b} \mathbb{A}_{S^\otimes}^{0|1} & \xrightarrow{\text{pr}_2^\otimes} & \mathbb{A}_{S^\otimes}^{0|1} \\ \sigma^U \times \text{id} \downarrow & & \downarrow \sigma_i^\otimes \\ U \times_{S_b} \mathbb{A}_{S^\otimes}^{0|1} & \xrightarrow{\pi^{\otimes\log} \circ (\eta^\otimes)^{-1}} & X^\otimes. \end{array}$$

is commutative and cartesian, where pr_2^\otimes denotes the projection to the second factor. We shall refer to such a collection of data \mathbb{U}^\star as a **pointed log twisted $(1|1)$ -chart** (around q) on $X^{\otimes\log}/S^{\otimes\log}$.

Proof. We may suppose, without loss of generality, that S^\otimes is affine. Let us take a log twisted $(1|1)$ -chart $(Y^{\otimes\log}, U^{\log}, \eta^{\otimes\log})$ around q on $X^{\otimes\log}/S^{\otimes\log}$ such that there is no nodal point in U . It follows from Proposition 1.2.5 that there exist

an étale scheme Σ^U over S_b and a closed immersion $\sigma^{Y^\circledast} : \Sigma^U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow Y^\circledast$ which make the diagram

$$(87) \quad \begin{array}{ccc} \Sigma^U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} & \xrightarrow{\text{pr}_2^\circledast} & \mathbb{A}_{S^\circledast}^{0|1} \\ \sigma^{Y^\circledast} \downarrow & & \downarrow \sigma_i^\circledast \\ Y^\circledast & \xrightarrow{\pi^{\circledast \log}} & X^\circledast \end{array}$$

commutate and cartesian. Consider the composite

$$(88) \quad \sigma_\eta^{Y^\circledast} : \Sigma^U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \xrightarrow{\sigma^{Y^\circledast}} Y^\circledast \xrightarrow{\eta^\circledast} U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}.$$

By applying Lemma 3.3.3 below, there exists an automorphism u^\circledast of $U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}$ over S^\circledast such that $u^\circledast \circ \sigma_\eta^{Y^\circledast} = \sigma^U \times \text{id}_{\mathbb{A}_{S^\circledast}^{0|1}}$ for some closed immersion $\sigma^U : \Sigma^U \rightarrow U$. Notice that (since there is no nodal point in U) the log structure of U^{\log} coincides with the pull-back from S_b^{\log} . Hence, u^\circledast extends to an automorphism $u^{\circledast \log}$ of $U^{\log} \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}$ over $S^{\circledast \log}$. Thus, the collection of data

$$(89) \quad \mathbb{U}^\star := (Y^{\circledast \log}, U^{\log}, u^{\circledast \log} \circ \eta^{\circledast \log}, \Sigma^U, \sigma^U)$$

obtained in this manner forms the desired collection. This completes the proof of Proposition 3.3.2. \square

The following lemma was used in the proof of Proposition 3.3.2.

Lemma 3.3.3.

Suppose that S^\circledast is affine. Let Σ be an affine scheme over S_b and U a smooth affine scheme over S_b . Then, for any closed immersion $\sigma^\circledast : \Sigma \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}$ over S^\circledast , there exists an automorphism u^\circledast of $U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}$ over S^\circledast such that the composite $u^\circledast \circ \sigma^\circledast : \Sigma \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}$ is of the form $\sigma_0 \times \text{id}_{\mathbb{A}_{S^\circledast}^{0|1}}$ for some closed immersion $\sigma_0 : \Sigma \rightarrow U$ over S_b .

Proof. In the following, let us construct two morphisms $\delta_1^\circledast : U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow U$ and $\delta_2^\circledast : U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow \mathbb{A}_{S^\circledast}^{0|1}$.

First, we shall consider δ_2^\circledast . The map $\Gamma(U, \mathcal{O}_{(U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1})_f}) \rightarrow \Gamma(\Sigma, \mathcal{O}_{(\Sigma \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1})_f})$ induced by the closed immersion σ^\circledast is surjective. Hence, by Proposition 1.4.3, the map

$$(90) \quad \begin{array}{ccc} \text{Hom}_{S^\circledast}(U \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}, \mathbb{A}_{S^\circledast}^{0|1}) & \rightarrow & \text{Hom}_{S^\circledast}(\Sigma \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1}, \mathbb{A}_{S^\circledast}^{0|1}) \\ h^\circledast & \mapsto & h^\circledast \circ \sigma^\circledast \end{array}$$

obtained by composing with σ^\circledast is surjective. Then, let us take δ_2^\circledast to be an inverse image of the projection $\Sigma \times_{S_b} \mathbb{A}_{S^\circledast}^{0|1} \rightarrow \mathbb{A}_{S^\circledast}^{0|1}$ via the surjection (90).

Next, we shall consider $\delta_1^{\mathbb{S}}$. We shall write

$$(91) \quad \sigma_1^{\mathbb{S}} : \Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1} \xrightarrow{\sigma^{\mathbb{S}}} U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1} \xrightarrow{\text{pr}^{\mathbb{S}}} U,$$

where the second arrow $\text{pr}^{\mathbb{S}}$ denotes the projection to the first factor. Also, write

$$(92) \quad \tilde{\sigma}_t^{\mathbb{S}} : \Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1} \rightarrow \Sigma \xrightarrow{\sigma_t^{\mathbb{S}}} U,$$

where the first arrow denotes the projection $\Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1} \rightarrow \Sigma$ to the first factor. Moreover, denote by \mathcal{J}_{Σ} and \mathcal{J}_U the (square nilpotent) ideal of $\mathcal{O}_{(\Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1})_b}$ and $\mathcal{O}_{(U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1})_b}$ corresponding to the closed immersions

$$(93) \quad \gamma_{\Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1}} : \Sigma \rightarrow (\Sigma \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1})_b \quad \text{and} \quad \gamma_{U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1}} : U \rightarrow (U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1})_b$$

respectively. Then, there exists an element $\partial \in \Gamma(\Sigma, \sigma_t^*(\mathcal{T}_{U/S_b}) \otimes \mathcal{J}_{\Sigma})$ such that $(\sigma_1)_b \boxplus^{\dagger} \partial = (\tilde{\sigma}_t^{\mathbb{S}})_b$ (cf. Proposition 2.4.3 (ii) for the definition of “ \boxplus^{\dagger} ”). Since the morphism

$$(94) \quad \Gamma(U, \mathcal{T}_{U/S_b} \otimes \mathcal{J}_U) \rightarrow \Gamma(\Sigma, \sigma_t^*(\mathcal{T}_{U/S_b}) \otimes \mathcal{J}_{\Sigma})$$

induced by σ_t is surjective, we obtain an inverse image $\tilde{\partial} \in \Gamma(U, \mathcal{T}_{U/S_b} \otimes \mathcal{J}_U)$ of ∂ via (94). Thus, we obtain a morphism

$$(95) \quad \delta_1^{\mathbb{S}} := (\text{pr}_b \boxplus^{\dagger} \tilde{\partial}) \circ \beta_{U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1}}^{\mathbb{S}} : U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1} \rightarrow U$$

over S_b .

It follows from the definitions of $\delta_1^{\mathbb{S}}$ and $\delta_2^{\mathbb{S}}$ that the endomorphism $u^{\mathbb{S}} := (\delta_1^{\mathbb{S}}, \delta_2^{\mathbb{S}})$ of $U \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1}$ turns out to be the desired automorphism. This completes the proof of Lemma 3.3.3. \square

Remark 3.3.4.

Let $\mathfrak{X}^{\mathbb{S}\star} := (X^{\mathbb{S}\log}/S^{\mathbb{S}\log}, \{[\sigma_i^{\mathbb{S}}]\}_{i=1}^r)$ be an r -pointed log twisted $(1|1)$ -curve of genus g over $S^{\mathbb{S}\log}$, q a geometric point of X_b , and $\mathbb{U} := (Y^{\mathbb{S}\log}, U^{\log}, \eta^{\mathbb{S}\log})$ a log twisted $(1|1)$ -chart around q on $X^{\mathbb{S}\log}/S^{\mathbb{S}\log}$. Suppose that we are given a strict superétale morphism $\pi_Y^{\mathbb{S}\log} : Y'^{\mathbb{S}\log} \rightarrow Y^{\mathbb{S}\log}$ such that $Y'^{\mathbb{S}\log}$ is affine and the image of the composite $Y'^{\mathbb{S}\log} \xrightarrow{\pi_Y^{\mathbb{S}\log}} Y^{\mathbb{S}\log} \rightarrow X^{\mathbb{S}\log}$ contains q . Then, by Proposition 1.2.5, there exist a strict étale morphism $\pi_U^{\log} : U'^{\log} \rightarrow U^{\log}$ and an isomorphism $\eta'^{\mathbb{S}\log} : Y'^{\mathbb{S}\log} \xrightarrow{\sim} U'^{\log} \times_{S_b} \mathbb{A}_{S^{\mathbb{S}}}^{0|1}$ over $S^{\mathbb{S}\log}$ (hence U' is affine)

which make the square diagram

$$(96) \quad \begin{array}{ccc} Y'^{\otimes \log} & \xrightarrow{\eta'^{\otimes \log}} & U'^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1} \\ \pi_Y^{\otimes \log} \downarrow & & \downarrow \pi_U^{\log} \times \text{id} \\ Y^{\otimes \log} & \xrightarrow{\eta^{\otimes \log}} & U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1} \end{array}$$

commute and cartesian. (Such a pair $(U'^{\log}, \eta'^{\otimes \log})$ is uniquely determined up to isomorphism). We shall write

$$(97) \quad \mathbb{U}|_{Y'^{\otimes \log}} := (Y'^{\otimes \log}, U'^{\log}, \eta'^{\otimes \log}),$$

which forms a log twisted $(1|1)$ -chart around q on $X^{\otimes \log}/S^{\otimes \log}$.

Suppose further that $q \in \text{Im}((\sigma_i)_b)$ and we are given Σ^U and σ^U as in (85) for which the collection of data $\mathbb{U}^\star := (Y^{\otimes \log}, U^{\log}, \eta^{\otimes \log}, \Sigma^U, \sigma^U)$ forms a pointed log twisted $(1|1)$ -chart around q on $X^{\otimes \log}/S^{\otimes \log}$. Let us write

$$(98) \quad \Sigma^{U'} := \Sigma^U \times_{\sigma^U, U, \pi_U} U' \quad \text{and} \quad \sigma^{U'} := \sigma^U \times \text{id}_{U'} : \Sigma^{U'} \rightarrow U'.$$

Then,

$$(99) \quad \mathbb{U}^\star|_{Y'^{\otimes \log}} := (Y'^{\otimes \log}, U'^{\log}, \eta'^{\otimes \log}, \Sigma^{U'}, \pi^{U'}, \sigma^{U'})$$

forms a pointed log twisted $(1|1)$ -chart \mathbb{U}^\star around q of $X^{\otimes \log}/S^{\otimes \log}$.

3.4. Superconformal structures.

Let us fix an r -pointed log twisted $(1|1)$ -curve $\mathfrak{X}^{\otimes \star} := (X^{\otimes \log}/S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r)$ of genus g over $S^{\otimes \log}$. We shall construct a new log structure on X^{\otimes} as follows. The ideal sheaf $\mathcal{I}_i \subseteq \mathcal{O}_{X_b}$ ($i = 1, \dots, r$) defining the closed immersion $(\sigma_i)_b$ is, by Proposition 3.3.2, an invertible sheaf. As explained in [13], Complement 1, it corresponds to a log structure $\alpha_{X_b}^{\sigma_i} : \mathcal{M}_{X_b}^{\sigma_i} \rightarrow \mathcal{O}_{X_b}$. We shall write

$$(100) \quad \mathcal{M}_{X_b}^\star := \mathcal{M}_{X_b} \oplus_{\mathcal{O}_{X_b}^\times} (\mathcal{M}_{X_b}^{\sigma_1} \oplus_{\mathcal{O}_{X_b}^\times} \cdots \oplus_{\mathcal{O}_{X_b}^\times} \mathcal{M}_{X_b}^{\sigma_r})$$

and define a log structure $\alpha_{X_b}^\star$ to be the amalgam

$$(101) \quad \alpha_{X_b}^\star := (\alpha_{X_b}, (\alpha_{X_b}^{\sigma_1}, \dots, \alpha_{X_b}^{\sigma_r})) : \mathcal{M}_{X_b}^\star \rightarrow \mathcal{O}_{X_b}.$$

We shall denote by

$$(102) \quad X^{\otimes \star \text{-log}} := (X^{\otimes}, \alpha_{X_b}^\star)$$

the resulting log superstack over $S^{\otimes \log}$, which admits a natural morphism $X^{\otimes \star \text{-log}} \rightarrow X^{\otimes \log}$. If $f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}$ denotes the structure morphism of $X^{\otimes \log}/S^{\otimes \log}$, then we shall write

$$(103) \quad f^{\otimes \star \text{-log}} : X^{\otimes \star \text{-log}} \rightarrow S^{\otimes \log}$$

for the composite of $f^{\otimes \log}$ with $X^{\otimes \star - \log} \rightarrow X^{\otimes \log}$. One verifies that $X^{\otimes \star - \log}/S^{\otimes \log}$ is log supersmooth of relative superdimension $1|1$. In particular, the $\mathcal{O}_{X^{\otimes \star - \log}/S^{\otimes \log}}$ (as well as $\Omega_{X^{\otimes \star - \log}/S^{\otimes \log}}$) is a supervector bundle of superrank $1|1$ (cf. Proposition 2.4.3 (i)).

Definition 3.4.1.

- (i) Let $X^{\otimes \log}$ be a log supersmooth superscheme over $S^{\otimes \log}$ of relative superdimension $1|1$. A **superconformal structure** on $X^{\otimes \log}/S^{\otimes \log}$ is a subsupervector bundle \mathcal{D} of superrank $0|1$ of $\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}$ (i.e., $\mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}/\mathcal{D}$ is a supervector bundle of superrank $1|0$) such that the $\mathcal{O}_{X^{\otimes \log}}$ -linear morphism

$$(104) \quad (\mathcal{D}^{\otimes 2} :=) \mathcal{D} \otimes_{\mathcal{O}_{X^{\otimes \log}}} \mathcal{D} \rightarrow \mathcal{T}_{X^{\otimes \log}/S^{\otimes \log}}/\mathcal{D}$$

$$\partial_1 \otimes \partial_2 \mapsto \frac{1}{2} \cdot \overline{[\partial_1, \partial_2]}$$

(where ∂_1 and ∂_2 are local sections of \mathcal{D}) is an isomorphism.

- (ii) An **r -pointed log twisted SUSY₁ curve of genus g** over $S^{\otimes \log}$ is a collection of data

$$(105) \quad {}_{\S 1} \mathfrak{Y}^{\otimes \star} := (Y^{\otimes \log}/S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$$

consisting of an r -pointed log twisted $(1|1)$ -curve $(Y^{\otimes \log}/S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r)$ of genus g over $S^{\otimes \log}$ and a superconformal structure \mathcal{D} on $Y^{\otimes \star - \log}/S^{\otimes \log}$.

Definition 3.4.2.

For $j \in \{1, 2\}$, let $S_j^{\otimes \log}$ be an fs log superscheme and ${}_{\S 1} \mathfrak{X}_j^{\otimes \star} := (f_j^{\otimes \log} : X_j^{\otimes \log} \rightarrow S_j^{\otimes \log}, \{[\sigma_{j,i}^{\otimes}]\}_{i=1}^r, \mathcal{D}_j)$ an r -pointed log twisted SUSY₁ curve of genus g over $S_j^{\otimes \log}$.

- (i) A **superconformal morphism** from ${}_{\S 1} \mathfrak{X}_1^{\otimes \star}$ to ${}_{\S 1} \mathfrak{X}_2^{\otimes \star}$ is a pair

$$(106) \quad {}_{\S 1} \Phi^{\otimes \star} := (\Phi^{\otimes \log}, \phi^{\otimes \log})$$

consisting of two morphisms $\Phi^{\otimes \log} : X_1^{\otimes \log} \rightarrow X_2^{\otimes \log}$, $\phi^{\otimes \log} : S_1^{\otimes \log} \rightarrow S_2^{\otimes \log}$ such that

- the square diagram

$$(107) \quad \begin{array}{ccc} X_1^{\otimes \log} & \xrightarrow{\Phi^{\otimes \log}} & X_2^{\otimes \log} \\ f_1^{\otimes \log} \downarrow & & \downarrow f_2^{\otimes \log} \\ S_1^{\otimes \log} & \xrightarrow{\phi^{\otimes \log}} & S_2^{\otimes \log} \end{array}$$

($i = 1, \dots, r$) is commutative and cartesian;

- $[\sigma_{1,i}^{\otimes}] = \Phi^{\otimes*}([\sigma_{2,i}^{\otimes}])$ (for any $i \in \{1, \dots, r\}$) and $\mathcal{D}_1 = \Phi^{\otimes*}(\mathcal{D}_2)$ via the isomorphism $\mathcal{T}_{X_1^{\otimes\bullet-\log}/S_1^{\otimes\log}} \xrightarrow{\sim} \Phi^{\otimes*}(\mathcal{T}_{X_2^{\otimes\bullet-\log}/S_2^{\otimes\log}})$ induced by $\Phi^{\otimes\log}$.
- (ii) Suppose further that $S_1^{\otimes\log} = S_2^{\otimes\log} (= S^{\otimes\log})$. A **superconformal isomorphism over $S^{\otimes\log}$** from ${}^{\S_1}\mathfrak{X}_1^{\otimes\bullet}$ to ${}^{\S_1}\mathfrak{X}_2^{\otimes\bullet}$ is a superconformal morphism ${}^{\S_1}\Phi^{\otimes\bullet} := (\Phi^{\otimes\log}, \phi^{\otimes\log}) : {}^{\S_1}\mathfrak{X}_1^{\otimes\bullet} \rightarrow {}^{\S_1}\mathfrak{X}_2^{\otimes\bullet}$ such that $\phi^{\otimes\log} = \text{id}_{S^{\otimes\log}}$ and $\Phi^{\otimes\log}$ is an isomorphism.

In the following Proposition 3.4.3, we discuss an explicit description of a superconformal structure (cf. e.g., [7], Lemma 3.1, for the case of smooth SUSY₁ curves over \mathbb{C} , i.e., super Riemann surfaces.)

Proposition 3.4.3.

Let U^{\log} be a log smooth scheme over S_b^{\log} of relative dimension 1. (In particular, $Z^{\otimes\log} := U^{\log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1}$ is a log supersmooth superscheme over $S^{\otimes\log}$ of relative superdimension 1|1). Suppose that we are given an element $z \in \Gamma(U, \mathcal{M}_U)$ such that $\Omega_{U^{\log}/S_b^{\log}} \cong \mathcal{O}_U \cdot d\log(z)$. Let us regard $d\log(z)$ and $d(\psi)$ as sections of $\Omega_{Z^{\otimes\log}/S^{\otimes\log}}$ via the projections $Z^{\otimes\log} \rightarrow U^{\log}$ and $Z^{\otimes\log} \rightarrow \mathbb{A}_{S^{\otimes}}^{0|1}$ respectively; these sections give a decomposition $\Omega_{Z^{\otimes\log}/S^{\otimes\log}} \cong \mathcal{O}_{Z^{\otimes}} \cdot d(\psi) \oplus \mathcal{O}_{Z^{\otimes}} \cdot d\log(z)$. In particular, we have

$$(108) \quad \mathcal{T}_{Z^{\otimes\log}/S^{\otimes\log}} \cong \mathcal{O}_{Z^{\otimes}} \cdot \partial_{\psi} \oplus \mathcal{O}_{Z^{\otimes}} \cdot \partial_z,$$

where $\{\partial_{\psi}, \partial_z\}$ is the dual basis of $\{d(\psi), d\log(z)\}$. Then, the following assertions are satisfied.

- (i) For each $a \in \Gamma(Z_b, \mathcal{O}_{Z_b}^{\times})$, the subsupermodule

$$(109) \quad \mathcal{D}_a := \mathcal{O}_{Z^{\otimes}} \cdot (\partial_{\psi} + a\psi \cdot \partial_z)$$

of $\mathcal{T}_{Z^{\otimes\log}/S^{\otimes\log}}$ forms a superconformal structure on $Z^{\otimes\log}/S^{\otimes\log}$. Moreover, the assignment $a \mapsto \mathcal{D}_a$ determines a bijection between the set $\Gamma(Z_b, \mathcal{O}_{Z_b}^{\times})$ and the set of superconformal structures on $Z^{\otimes\log}/S^{\otimes\log}$.

- (ii) Let us take two superconformal structures on $Z^{\otimes\log}/S^{\otimes\log}$ of the form $\mathcal{D}_a, \mathcal{D}_b$ for some $a, b \in \Gamma(Z_b, \mathcal{O}_{Z_b}^{\times})$. Suppose that there exists an element $c \in \Gamma(Z_b, \mathcal{O}_{Z_b}^{\times})$ such that $c^2 \cdot a = b$. (According to Proposition 1.2.5, such an element c exists after possibly replacing U with its étale covering.) If we write ι_c for the automorphism of $Z^{\otimes\log}$ over $U^{\log} \times_{S_b} S^{\otimes}$ given by assigning $\psi \mapsto c \cdot \psi$, then the isomorphism $\mathcal{T}_{Z^{\otimes\log}/S^{\otimes\log}} \xrightarrow{\sim} \iota_c^*(\mathcal{T}_{Z^{\otimes\log}/S^{\otimes\log}})$ induces an isomorphism $\mathcal{D}_a \xrightarrow{\sim} \iota_c^*(\mathcal{D}_b)$.

Proof. First, we consider assertion (i). For each $a \in \Gamma(Z_b, \mathcal{O}_{Z_b}^\times)$, we have

$$\begin{aligned}
 (110) \quad & \frac{1}{2} \cdot [\partial_\psi + a\psi \cdot \partial_z, \partial_\psi + a\psi \cdot \partial_z] \\
 &= \partial_\psi^2 + ((a\psi \cdot \partial_z) \circ \partial_\psi + \partial_\psi \circ (a\psi \cdot \partial_z)) + (a\psi \cdot \partial_z)^2 \\
 &= 0 + a \cdot \partial_z + 0 \\
 &= a \cdot \partial_z.
 \end{aligned}$$

Hence, (since the sections $\{\partial_\psi + a\psi \cdot \partial_z, a \cdot \partial_z\}$ generate $\mathcal{T}_{Z^{\otimes \log}/S^{\otimes \log}} \mathcal{D}_a$ forms a superconformal structure on $Z^{\otimes \log}/S^{\otimes \log}$.

Next, we shall consider the bijectivity of the assignment $a \mapsto \mathcal{D}_a$. Let \mathcal{D} be a superconformal structure on $Z^{\otimes \log}/S^{\otimes \log}$. There exists an open covering $\{Z_\gamma\}_\gamma$ of Z_b such that each restriction $\mathcal{D}|_{Z_\gamma}$ may be generated by some $\partial_\gamma \in \Gamma(Z_\gamma, \mathcal{D})$. The section ∂_γ may be described as $\partial_\gamma := a_\gamma \psi \cdot \partial_z + b_\gamma \cdot \partial_\psi$ (where $a_\gamma, b_\gamma \in \Gamma(Z_\gamma, \mathcal{O}_{X_b})$). Then,

$$(111) \quad \partial_\gamma^2 := \frac{1}{2} \cdot [\partial_\gamma, \partial_\gamma] = a_\gamma b_\gamma \cdot \partial_z + a_\gamma (\partial_z(b_\gamma)) \psi \cdot \partial_\psi.$$

Since $\{\partial_\gamma, \partial_\gamma^2\}$ generates $\mathcal{T}_{Z^{\otimes \log}/S^{\otimes \log}}|_{Z_\gamma}$, both a_γ and b_γ lie in $\Gamma(Z_\gamma, \mathcal{O}_{Z_b}^\times)$. Thus, there exists *uniquely* an element ∂'_γ (i.e., $\partial'_\gamma := b_\gamma^{-1} \cdot \partial_\gamma$) in $\Gamma(Z_\gamma, \mathcal{D})$ of the form $\partial_\psi + a'_\gamma \psi \cdot \partial_z$ (for some $a'_\gamma \in \Gamma(Z_\gamma, \mathcal{O}_{Z_b}^\times)$). In particular, $\{\partial'_\gamma\}_\gamma$ may be glued together to an element of $\Gamma(Z_b, \mathcal{D})$ of the form $\partial_\psi + a' \psi \cdot \partial_z$ (for a unique $a' \in \Gamma(Z_b, \mathcal{O}_{Z_b}^\times)$). This assignment $\mathcal{D} \mapsto a'$ determines an inverse to the assignment $a \mapsto \mathcal{D}_a$. Consequently, $a \mapsto \mathcal{D}_a$ is bijective, as desired.

Finally, assertion (ii) follows immediately from the definition of ι_c . \square

In particular, we have the following assertion.

Corollary 3.4.4.

Let ${}^{\S_1} \mathfrak{X}^{\otimes \star} := (X^{\otimes \log}/S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$ be an r -pointed log twisted $SUSY_1$ curve of genus g over $S^{\otimes \log}$. Then, there exists a collection of data

$$(112) \quad \{(Y_\gamma^{\otimes \log} \xrightarrow{\pi_\gamma^{\otimes \log}} X^{\otimes \log}, U_\gamma^{\otimes \log}, \eta_\gamma^{\otimes \log}, z_\gamma)\}_\gamma,$$

where

- $\{(Y_\gamma^{\otimes \log} \xrightarrow{\pi_\gamma^{\otimes \log}} X^{\otimes \log}, U_\gamma^{\otimes \log}, \eta_\gamma^{\otimes \log})\}_\gamma$ is a collection of log twisted $(1|1)$ -chart on $X^{\otimes \log}/S^{\otimes \log}$ such that $\coprod_\gamma Y_\gamma^{\otimes \log} \rightarrow X^{\otimes \log}$ is a strict superétale covering of $X^{\otimes \log}$;
- Each z_γ is an element of $\Gamma(U_\gamma, \mathcal{M}_{U_\gamma})$ such that $d\log(z_\gamma)$ generates $\Omega_{U_\gamma^{\otimes \log}/S_b^{\otimes \log}}$ and the superconformal structure $\mathcal{D}|_{Y_\gamma^{\otimes \log}}$ on $Y_\gamma^{\otimes \log}/S^{\otimes \log}$ obtained by restricting \mathcal{D} to $Y_\gamma^{\otimes \log}$ coincides with

$$(113) \quad \mathcal{O}_{U_\gamma^{\otimes \log} \times_{S_b} \mathbb{A}_{S^{\otimes}}} \cdot (\partial_\psi + \psi \cdot \partial_{z_\gamma}) \subseteq \mathcal{T}_{U_\gamma^{\otimes \log} \times_{S_b} \mathbb{A}_{S^{\otimes}}^{0|1}/S^{\otimes \log}}$$

(where $\{\partial_\psi, \partial_{z_\gamma}\}$ is the dual basis of $\{d(\psi), d\log(z_\gamma)\}$) via the isomorphism $\mathcal{T}_{Z^{\otimes \log}/S^{\otimes \log}} \xrightarrow{\sim} (\eta_\gamma^{\otimes})^*(\mathcal{T}_{U_\gamma^{\log} \times_{S_0} \mathbb{A}_{S^{\otimes}}^{0|1}/S^{\otimes \log}})$ induced by η_γ^{\otimes} .

3.5. Kodaira-Spencer morphisms.

Let ${}_{\S_1}\mathfrak{X}^{\otimes \star} := (f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$ be an r -pointed log twisted SUSY₁ curve of genus g over $S^{\otimes \log}$. Let us define an $f_b^{-1}(\mathcal{O}_{S^{\otimes}})$ -subsupermodule $\mathcal{T}_{X^{\otimes \star}\text{-log}/S_0}^{\mathcal{D}}$ (resp., $\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}}$) of $\mathcal{T}_{X^{\otimes \star}\text{-log}/S_0}$ (resp., $\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}$) to be

$$(114) \quad \mathcal{T}_{X^{\otimes \star}\text{-log}/S_0}^{\mathcal{D}} := \{\partial \in \mathcal{T}_{X^{\otimes \star}\text{-log}/S_0} \mid [\partial, \mathcal{D}] \subseteq \mathcal{D}\} \\ (\text{resp., } \mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}} := \mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}} \cap \mathcal{T}_{X^{\otimes \star}\text{-log}/S_0}^{\mathcal{D}}).$$

Since $X^{\otimes \star\text{-log}}$ is log supersmooth over $S^{\otimes \log}$, the dual of the sequence (45) gives rise to a short exact sequence of $\mathcal{O}_{X^{\otimes}}$ -supermodules:

$$(115) \quad 0 \rightarrow \mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}} \rightarrow \mathcal{T}_{X^{\otimes \star}\text{-log}/S_0}^{\mathcal{D}} \rightarrow f^{\otimes *}(\mathcal{T}_{S^{\otimes \log}/S_0}) \rightarrow 0.$$

(Here, the pulled-back $\mathcal{O}_{S^{\otimes}}$ -supermodule $f^{\otimes *}(-)$ via f^{\otimes} defined preceding Definition 1.1.3 may be also defined in our situation, i.e., X^{\otimes} is a superstack.) The higher direct image $\mathbb{R}^1 f_{b*}(\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}})$ admits naturally a structure of $\mathcal{O}_{S^{\otimes}}$ -supermodule. Denote by $\mathbb{R}^1 f_*^{\otimes}(\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}})$ the resulting $\mathcal{O}_{S^{\otimes}}$ -supermodule. The connecting homomorphism of (115) yields an $\mathcal{O}_{S^{\otimes}}$ -linear morphism

$$(116) \quad \mathcal{KS}({}_{\S_1}\mathfrak{X}^{\otimes \star}) : \mathcal{T}_{S^{\otimes \log}/S_0} \rightarrow \mathbb{R}^1 f_*^{\otimes}(\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}}),$$

which is referred to as the **Kodaira-Spencer morphism** of ${}_{\S_1}\mathfrak{X}^{\otimes \star}$.

The following proposition will be used in the discussion in Remark 4.3.1 and Proposition 5.1.1.

Proposition 3.5.1.

There exists a canonical $f_b^{-1}(\mathcal{O}_{S^{\otimes}})$ -linear isomorphism $\mathcal{T}_{X^{\otimes \star}\text{-log}/S^{\otimes \log}}^{\mathcal{D}} \xrightarrow{\sim} \mathcal{D}^{\otimes 2}$.

Proof. The assertion follows from an argument similar to the argument in the proof of [15], Lemma 2.1, together with Corollary 3.4.4 of the present paper. \square

3.6. Stable log twisted SUSY₁ curves.

Let λ be an even positive integer invertible in S_0 . Let us recall from [5], Definition 4.1.3 and Remark 4.2.6, the notion of a λ -stable twisted curve. We shall write

$$(117) \quad {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$$

for the moduli stack classifying r -pointed λ -stable twisted curves over S_0 of genus g . It is a geometrically connected, proper, and smooth Deligne-Mumford

stack over S_0 of relative dimension $3g - 3 + r$ (cf. [5], Corollary 4.2.8). Denote by $(\mathfrak{C}, \{[\sigma_{\mathfrak{C},i}]\}_{i=1}^r)$ the tautological r -pointed λ -stable twisted curve over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$. Both ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$ and \mathfrak{C} admit canonically log structures (cf. [19], Theorem 1.9). If ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$ and \mathfrak{C}^{\log} denote the resulting log stacks, then the structure morphism of \mathfrak{C} over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$ extends to a log smooth morphism $\mathfrak{C}^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$.

Let $s^{\log} : \underline{S}^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$ be a morphism whose underlying morphism of stacks classifies an r -pointed twisted curve $\underline{X}^{\star} := (\underline{X}/\underline{S}, \{[\sigma_i]\}_{i=1}^r)$ of genus g . Then, by equipping \underline{X} with the log structure pulled-back from $\mathfrak{C}^{\log} \times_{{}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}} \underline{S}^{\log}$ via the isomorphism $\underline{X} \xrightarrow{\sim} \mathfrak{C} \times_{{}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}} \underline{S}$ induced by s , we have a log stack

$$(118) \quad \underline{X}^{\star-\log}$$

together with a log smooth morphism $\underline{X}^{\star-\log} \rightarrow \underline{S}^{\log}$.

Moreover, let us write $\mathfrak{M}_{g,r}$ for the moduli stack classifying r -pointed proper *smooth* curves over S_0 of genus g . By the natural inclusion $\mathfrak{M}_{g,r} \hookrightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$, we may regard $\mathfrak{M}_{g,r}$ as a dense open substack of ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$. Also, this open locus of ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$ coincides with the locus in which the log structure of ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$ becomes trivial.

Definition 3.6.1.

A **stable log twisted SUSY₁ curve of type (g, r, λ)** over S^{Slog} is an r -pointed log twisted SUSY₁ curve of genus g over S^{Slog} whose underlying pointed twisted curve is λ -stable.

Let (g, r, λ) be a triple of nonnegative integers satisfying that $2g - 2 + r > 0$ and λ is even. Then, the stable log twisted SUSY₁ curves of type (g, r, λ) over log superschemes and superconformal morphisms between them form a category fibered in groupoids over $\mathfrak{Sch}_{/S_0}^{\text{Slog}}$:

$$(119) \quad \begin{aligned} {}_{\S 1} \overline{\mathfrak{M}}_{g,r,\lambda}^{\text{Slog}} &\rightarrow \mathfrak{Sch}_{/S_0}^{\text{Slog}} \\ {}_{\S 1} \mathfrak{X}^{\text{S}\star} (\text{over } S^{\text{Slog}}) &\mapsto S^{\text{Slog}}. \end{aligned}$$

One verifies from a standard argument in descent theory that ${}_{\S 1} \overline{\mathfrak{M}}_{g,r,\lambda}^{\text{Slog}}$ forms a stack with respect to the strict superétale pretopology in $\mathfrak{Sch}_{/S_0}^{\text{Slog}}$. We shall denote by

$$(120) \quad ({}_{\S 1} \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$$

the restriction of ${}_{\S 1} \overline{\mathfrak{M}}_{g,r,\lambda}^{\text{Slog}}$ to the full subcategory $\mathfrak{Sch}_{/S_0}^{\log} \subseteq \mathfrak{Sch}_{/S_0}^{\text{Slog}}$. The assignment from each stable log twisted SUSY₁ curve over an fs log scheme

to its underlying pointed twisted curve determines a morphism $(\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$; it extends to a morphism

$$(121) \quad (\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$$

of log stacks.

4. SUPERCONFORMAL STRUCTURE V.S. SPIN STRUCTURE

This section is devoted to understand the structure of the reduced stack $(\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$ of $\S_1 \overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$. The point is that to giving a pointed log twisted SUSY₁ curve over a log scheme is, via a natural procedure, equivalent to giving a pointed log twisted curve equipped with an additional data called a pointed spin structure (cf. Definition 4.1.1). Thus, if ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ (cf. (124)) denotes the moduli stack classifying λ -stable log twisted curves of type (g, r) equipped with a parabolic spin structure, then it is canonically isomorphic to $(\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$ of $\S_1 \overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$, as shown in Proposition 4.4.1.

In the following, *we suppose that r is even.*

4.1. Parabolic spin structures.

Let \underline{S}^{\log} be an fs log scheme and $\underline{X}^{\star} := (\underline{X}/\underline{S}, \{[\underline{\sigma}_i]\}_{i=1}^r)$ be an r -pointed twisted curve of genus g over the underlying scheme \underline{S} of \underline{S}^{\log} . Hence, by the discussion preceding Definition 3.6.1, we have a log smooth morphism $\underline{X}^{\star-\log} \rightarrow \underline{S}^{\log}$ and $\Omega_{\underline{X}^{\star-\log}/\underline{S}^{\log}}$ is a line bundle of total degree $2g - 2 + r$. Note that for each $i \in \{1, \dots, r\}$, there exists a canonical isomorphism

$$(122) \quad \Lambda_i : \underline{\sigma}_i^*(\Omega_{\underline{X}^{\star-\log}/\underline{S}^{\log}}) \xrightarrow{\sim} \mathcal{O}_{\underline{S}}$$

which maps any local section of the form $\underline{\sigma}_i^*(d\log(x))$ to $1 \in \mathcal{O}_{\underline{S}}$, where x is a local function defining the closed substack $[\underline{\sigma}_i]$ of \underline{X} . We shall write

$$(123) \quad \mathfrak{Spin}_{\underline{X}^{\star}}$$

for the groupoid defined as follows:

- The *objects* in $\mathfrak{Spin}_{\underline{X}^{\star}}$ are pairs (\mathcal{L}, η) , where \mathcal{L} denotes a line bundle on \underline{X} such that $\underline{\sigma}_i^*(\mathcal{L}) \cong \mathcal{O}_{\underline{S}}$ for any $i \in \{1, \dots, r\}$ and η denotes an isomorphism $\mathcal{L}^{\otimes 2} \xrightarrow{\sim} \Omega_{\underline{X}^{\star-\log}/\underline{S}^{\log}}$.
- The *morphisms* from (\mathcal{L}, η) to (\mathcal{L}', η') (where both (\mathcal{L}, η) and (\mathcal{L}', η') are objects in $\mathfrak{Spin}_{\underline{X}^{\star}}$) are isomorphisms $\iota : \mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ satisfying the equality $\eta_2 \circ \iota^{\otimes 2} = \eta_1$.

Definition 4.1.1.

We shall refer to such a pair (\mathcal{L}, η) (i.e., an object of $\mathfrak{Spin}_{\underline{\mathfrak{X}}^\star}$) as a **pointed spin structure** on $\underline{\mathfrak{X}}^\star$.

Remark 4.1.2.

Suppose that we are given a line bundle \mathcal{L}_0 on \underline{X} together with an isomorphism $\eta_0 : \mathcal{L}_0^{\otimes 2} \xrightarrow{\sim} \Omega_{\underline{X}^\star\text{-log}/\underline{S}^{\text{log}}}$. Since the composite $\Lambda_i \circ \underline{\sigma}_i^*(\eta_0)$ (for each $i \in \{1, \dots, r\}$) is an isomorphism $\underline{\sigma}_i^*(\mathcal{L})^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_{\underline{S}}$, the line bundle $\underline{\sigma}_i^*(\mathcal{L})$ defines a μ_2 -torsor over \underline{S} . Hence, after possibly base-changing $\underline{\mathfrak{X}}^\star$ via an étale covering $\underline{S}' \rightarrow \underline{S}$ of \underline{S} , the pair (\mathcal{L}_0, η_0) becomes a pointed spin structure on $\underline{\mathfrak{X}}^\star$. Indeed, if \underline{S}_i denotes the total space of the μ_2 -torsor corresponding to $\underline{\sigma}_i^*(\mathcal{L})$, then it suffices to choose the étale covering $\underline{S}' = \underline{S}_1 \times_{\underline{S}} \underline{S}_2 \times_{\underline{S}} \dots \times_{\underline{S}} \underline{S}_r$ of \underline{S} .

Denote by

$$(124) \quad {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$$

the category fibered in groupoids over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r}$ whose fiber over $\underline{S} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r}$ (where \underline{S} is a scheme) classifying an r -pointed λ -stable twisted curve $\underline{\mathfrak{X}}^\star$ is the groupoid $\mathfrak{Spin}_{\underline{\mathfrak{X}}^\star}$. One verifies from [5], Corollary 4.2.8 (and the fact that $\deg(\Omega_{\underline{X}^\star\text{-log}/\underline{S}^{\text{log}}}) = 2g - 2 + r$ is even) that ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$ may be represented by a smooth proper Deligne-Mumford stack over S_0 of relative dimension $3g - 3 + r$ and the forgetting morphism

$$(125) \quad {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}$$

is finite and étale. Indeed, according to the discussion in Remark 4.1.2, ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$ turns out to be finite and étale over the moduli stack classifying r -pointed λ -stable twisted curves $(\underline{X}/\underline{S}, \{[\underline{\sigma}_i]\}_{i=1}^r)$ of genus g equipped with a square root of $\Omega_{\underline{X}^\star\text{-log}/\underline{S}^{\text{log}}}$. We equip ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$ with the log structure pulled-back from ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\text{log}}$. Write

$$(126) \quad {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\text{log}}$$

for the resulting fs log stack (hence, (125) extends to ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\text{log}} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\text{log}}$).

4.2. From $({}^{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\text{log}}$ to ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\text{log}}$

The main goal of this section is to prove Proposition 4.4.1 described at the end of this section, i.e., to construct an equivalence of categories $({}^{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\text{log}} \xrightarrow{\sim} {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\text{log}}$. To this end, we construct first a morphism $({}^{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\text{log}} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\text{log}}$ over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\text{log}}$.

Let \underline{S}^{\log} be an fs log scheme and $\underline{S}^{\log} \rightarrow (\S_1 \overline{\mathcal{M}}_{g,r,\lambda})_t^{\log}$ a morphism classifying a stable log twisted SUSY₁ curve

$$(127) \quad \S_1 \mathfrak{X}^{\otimes \star} := (f^{\otimes \log} : X^{\otimes \log} \rightarrow \underline{S}^{\log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$$

of type (g, r, λ) over \underline{S}^{\log} . The morphism $\gamma_X : X_t \rightarrow X_b$ is an isomorphism, and allows us to identify the r -pointed λ -stable twisted curve

$$(128) \quad \mathfrak{X}_b^{\star} := (f_b : X_b \rightarrow \underline{S}, \{[(\sigma_i)_b]\}_{i=1}^r)$$

with the underlying pointed twisted curve of $(X^{\otimes \log}/\underline{S}^{\log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r)$. One verifies that there exist a line bundle \mathcal{L}^{\star} on X_b and an isomorphism $\Upsilon : X^{\otimes} \xrightarrow{\sim} \langle X_b, \mathcal{L}^{\star} \rangle^{\otimes}$ over \underline{S} which sends $[\sigma_i^{\otimes}]$ (for each $i \in \{1, \dots, r\}$) to the closed subscheme of $\langle X_b, \mathcal{L}^{\star} \rangle^{\otimes}$ represented by the closed immersion $\langle \underline{S}, (\sigma_i)_b^*(\mathcal{L}^{\star}) \rangle \rightarrow \langle X_b, \mathcal{L}^{\star} \rangle^{\otimes}$ extending $(\sigma_i)_b$. In particular, we have

$$(129) \quad (\sigma_i)_b^*(\mathcal{L}^{\star}) \cong \mathcal{O}_{\underline{S}}.$$

The isomorphism Υ gives rise to an isomorphism

$$(130) \quad \Omega_{X^{\otimes \star \log}/\underline{S}^{\log}} \xrightarrow{\sim} (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \Omega_{X_b^{\star \log}/\underline{S}^{\log}}) \oplus (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star})$$

of $\mathcal{O}_{X^{\otimes}}$ -supermodules, where, in the right-hand side, the sections of the forms $(1 \otimes a, 0)$ and $(0, 1 \otimes b)$ (for some $a \in \Omega_{X_b^{\star \log}/\underline{S}^{\log}}$ and $b \in \mathcal{L}^{\star}$) are defined to be bosonic and fermionic sections respectively. Consider its dual

$$(131) \quad \mathcal{T}_{X^{\otimes \star \log}/\underline{S}^{\log}} \xrightarrow{\sim} (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{T}_{X_b^{\star \log}/\underline{S}^{\log}}) \oplus (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star \vee}).$$

It follows from Proposition 3.4.3 (i) that the composite morphism

$$(132) \quad \begin{aligned} \mathcal{D} &\hookrightarrow \mathcal{T}_{X^{\otimes \star \log}/\underline{S}^{\log}} \\ &\xrightarrow{(131)} (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{T}_{X_b^{\star \log}/\underline{S}^{\log}}) \oplus (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star \vee}) \\ &\rightarrow \mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star \vee} \end{aligned}$$

(where the third morphism denotes the projection to the second factor) between supervector bundles of superrank $0|1$ is surjective, and hence, an isomorphism. Moreover, we have a composite isomorphism

$$(133) \quad \begin{aligned} \mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{T}_{X_b^{\star \log}/\underline{S}^{\log}} &\rightarrow (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{T}_{X_b^{\star \log}/\underline{S}^{\log}}) \oplus (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star \vee}) \\ &\xrightarrow{(131)^{-1}} \mathcal{T}_{X^{\otimes \star \log}/\underline{S}^{\log}} \\ &\rightarrow \mathcal{T}_{X^{\otimes \star \log}/\underline{S}^{\log}}/\mathcal{D} \end{aligned}$$

(where the first morphism denotes the inclusion into the first factor). The isomorphism (104) in our situation may be described as an isomorphism

$$(134) \quad (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} (\mathcal{L}^{\star \vee})^{\otimes 2}) = (\mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{L}^{\star \vee})^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_{X^{\otimes}} \otimes_{\mathcal{O}_{X_b}} \mathcal{T}_{X_b^{\star \log}/\underline{S}^{\log}}$$

via the composite isomorphisms (132) and (133). The restriction of this isomorphism to the bosonic part (and taking its dual) becomes an isomorphism $\eta^\star : (\mathcal{L}^\star)^{\otimes 2} \xrightarrow{\sim} \Omega_{X_b^{\star-\log}/\underline{S}^{\log}}$. Thus, the pair $(\mathcal{L}^\star, \eta^\star)$ forms (thanks to (129)) a pointed spin structure on \mathfrak{X}_b^\star . If $\underline{S} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$ denotes the classifying morphism of $(\mathcal{L}^\star, \eta^\star)$, then it extends uniquely to a morphism $\underline{S}^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$. The assignment ${}_{\S 1}\mathfrak{X}^{\otimes \star} \mapsto (\mathcal{L}^\star, \eta^\star)$ is functorial with respect to \underline{S}^{\log} and hence, determines a morphism

$$(135) \quad ({}_{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$$

over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$.

4.3. From ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ to $({}_{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$.

Conversely, we shall construct a morphism ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log} \rightarrow ({}_{\S 1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$. Let \underline{S}^{\log} be an fs log scheme and $\underline{S}^{\log} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ a morphism whose underlying morphism $\underline{S} \rightarrow {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}$ classifies a spin structure (\mathcal{L}, η) on an r -pointed λ -stable twisted curve $\underline{X}^\star := (f : \underline{X} \rightarrow \underline{S}, \{[\underline{\sigma}_i]\}_{i=1}^r)$. (In particular, we have a morphism $\underline{S}^{\log} \rightarrow \underline{S}^{f-\log}$.) By fixing an isomorphism $\underline{\sigma}_i^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_{\underline{S}}$ (for each $i = 1, \dots, r$), we obtain a composite closed immersion

$$(136) \quad \underline{\sigma}_{(\mathcal{L}, \eta), i}^{\otimes} : \mathbb{A}_{\underline{S}}^{0|1} \xrightarrow{\sim} \langle \underline{S}, \underline{\sigma}_i^*(\mathcal{L}) \rangle^{\otimes} \rightarrow \langle \underline{X}, \mathcal{L} \rangle^{\otimes}$$

extending $\underline{\sigma}_i$. We shall write $\underline{X}^{\log} := \underline{X}^{f-\log} \times_{\underline{X}^{f-\log}} \underline{S}^{\log}$, and hence, obtain a log superstack $\langle \underline{X}, \mathcal{L} \rangle^{\otimes \log}$ (cf. (37)) over \underline{S}^{\log} . The collection of data

$$(137) \quad \underline{\mathfrak{X}}_{(\mathcal{L}, \eta)}^{\otimes \star} := (\langle \underline{X}, \mathcal{L} \rangle^{\otimes \log} / \underline{S}^{\log}, \{[\underline{\sigma}_{(\mathcal{L}, \eta), i}^{\otimes}]\}_{i=1}^r)$$

forms an r -pointed log twisted $(1|1)$ -curve of genus g over \underline{S}^{\log} . Since $\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}} \cong \mathcal{O}_{\underline{X}} \oplus \mathcal{L}$, we obtain (cf. (130)) a composite isomorphism

$$(138) \quad \begin{aligned} & \mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star-\log} / \underline{S}^{\log}} \\ & \xrightarrow{\sim} (\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}} \otimes_{\mathcal{O}_{\underline{X}}} \mathcal{T}_{\underline{X}^{\star-\log} / \underline{S}^{f-\log}}) \oplus (\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}} \otimes_{\mathcal{O}_{\underline{X}}} \mathcal{L}^\vee) \\ & \xrightarrow{\sim} \mathcal{T}_{\underline{X}^{\star-\log} / \underline{S}^{f-\log}} \oplus \mathcal{L}^\vee \oplus \mathcal{L}^\vee \oplus \mathcal{O}_{\underline{X}} \end{aligned}$$

of $\mathcal{O}_{\underline{X}}$ -modules. Consider the $\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}}$ -linear injection

$$(139) \quad (\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}} \otimes_{\mathcal{O}_{\underline{X}}} \mathcal{L}^\vee =:) \mathcal{L}^\vee \oplus \mathcal{O}_{\underline{X}} \hookrightarrow \mathcal{T}_{\underline{X}^{\star-\log} / \underline{S}^{f-\log}} \oplus \mathcal{L}^\vee \oplus \mathcal{L}^\vee \oplus \mathcal{O}_{\underline{X}} \\ (a, b) \mapsto (0, a, a, b).$$

Write $\mathcal{D}_{(\mathcal{L}, \eta)}$ for the subsupervector bundle (of superrank $0|1$) of $\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star-\log} / \underline{S}^{\log}}$ corresponding, via the composite isomorphism (138), to the image of (139).

Then, one verifies immediately that the collection of data

$$(140) \quad \S \mathfrak{X}_{(\mathcal{L}, \eta)}^{\otimes \star} := (\langle \underline{X}, \mathcal{L} \rangle^{\otimes \log} / \underline{S}^{\log}, \{[\underline{\mathcal{Q}}_{(\mathcal{L}, \eta), i}^{\otimes}] \}_{i=1}^r, \mathcal{D}_{(\mathcal{L}, \eta)})$$

forms a stable log twisted SUSY_1 curve of type (g, r, λ) over \underline{S}^{\log} whose underlying pointed twisted curve is isomorphic to \mathfrak{X}^{\star} . It determines a morphism $\underline{S}^{\log} \rightarrow (\S_1 \overline{\mathfrak{M}}_{g, r, \lambda})_t^{\log}$. By varying \underline{S}^{\log} with the various fs log schemes, we obtain a morphism

$$(141) \quad {}^{\text{tw}} \overline{\mathfrak{M}}_{g, r, \lambda, \text{spin}}^{\log} \rightarrow (\S_1 \overline{\mathfrak{M}}_{g, r, \lambda})_t^{\log}$$

over ${}^{\text{tw}} \overline{\mathfrak{M}}_{g, r, \lambda}^{\log}$.

Remark 4.3.1.

We keep the above notation. By applying Proposition 3.5.1, we have a composite isomorphism of $\underline{f}^{-1}(\mathcal{O}_{\underline{S}})$ -modules:

$$(142) \quad \begin{aligned} \mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)} &\xrightarrow{\sim} (\mathcal{D}_{(\mathcal{L}, \eta)})^{\otimes 2} \\ &\xrightarrow{\sim} \mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}} \otimes_{\mathcal{O}_{\underline{X}}} (\mathcal{L}^{\vee})^{\otimes 2} \\ &\xrightarrow{\sim} \mathcal{T}_{\underline{X}^{\star \log} / \underline{S}^{f \log}} \oplus \mathcal{L}^{\vee}, \end{aligned}$$

where the second isomorphism follows from the definition of $\mathcal{D}_{(\mathcal{L}, \eta)}$ (cf. (139)).

Let us equip $\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)}$ with a structure of $\mathcal{O}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes}}$ -supermodule via the first isomorphism in (142). Then, (142) induces two isomorphisms of $\mathcal{O}_{\underline{X}}$ -modules

$$(143) \quad (\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)})_b \cong \mathcal{T}_{\underline{X}^{\star \log} / \underline{S}^{f \log}}, \quad (\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)})_f \cong \mathcal{L}^{\vee}.$$

In particular, we have

$$(144) \quad \deg((\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)})_b) = -2g + 2 - r, \quad \deg((\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)})_f) = -g + 1 - \frac{r}{2}.$$

The inclusion $\mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}^{\mathcal{D}(\mathcal{L}, \eta)} \hookrightarrow \mathcal{T}_{\langle \underline{X}, \mathcal{L} \rangle^{\otimes \star \log} / \underline{S}^{\log}}$ corresponds, via the isomorphisms (138) and (142), to the inclusion

$$(145) \quad \begin{aligned} \mathcal{T}_{\underline{X}^{\star \log} / \underline{S}^{f \log}} \oplus \mathcal{L}^{\vee} &\rightarrow \mathcal{T}_{\underline{X}^{\star \log} / \underline{S}^{f \log}} \oplus \mathcal{L}^{\vee} \oplus \mathcal{L}^{\vee} \oplus \mathcal{O}_{\underline{X}} \\ (a, b) &\mapsto (a, b, b, 0). \end{aligned}$$

Proposition 4.3.2.

Let $\S_1 \mathfrak{X}^{\otimes \star} := (f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$ be a stable log twisted SUSY_1 curve of type (g, r, λ) over $S^{\otimes \log}$. Then, $\mathbb{R}^2 f_*^{\otimes}(\mathcal{T}_{X^{\otimes \star \log} / S^{\otimes \log}}^{\mathcal{D}}) = 0$. Also, the $\mathcal{O}_{S^{\otimes}}$ -supermodule $\mathbb{R}^1 f_*^{\otimes}(\mathcal{T}_{X^{\otimes \star \log} / S^{\otimes \log}}^{\mathcal{D}})$ is locally free of rank $3g - 3 + r|2g - 2 + \frac{r}{2}|$ and the formulation of $\mathbb{R}^1 f_*^{\otimes}(\mathcal{T}_{X^{\otimes \star \log} / S^{\otimes \log}}^{\mathcal{D}})$ commutes with base-change

with respect to S^\circledcirc . Moreover, if S^\circledcirc is affine (i.e., $S^\circledcirc = S_b$) and \mathcal{F} is a quasi-coherent \mathcal{O}_{S_b} -module, then the natural morphism

$$(146) \quad \mathbb{R}^1 f_{b*}((\mathcal{T}_{X^\circledcirc \star \text{-log}/S_b^{\text{log}}}^\mathcal{D})_{(-)}) \otimes \mathcal{F} \rightarrow \mathbb{R}^1 f_{b*}((\mathcal{T}_{X^\circledcirc \star \text{-log}/S_b^{\text{log}}}^\mathcal{D} \otimes_{f_b^{-1}(\mathcal{O}_{S_b})} f_b^{-1}(\mathcal{F}))_{(-)}),$$

where $(-)$ denotes either “ \flat ” or “ f ”, is an isomorphism.

Proof. In the following, we consider $\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D}$ as being equipped with a structure of $\mathcal{O}_{X^\circledcirc}$ -supermodule by transposing the structure of $\mathcal{O}_{X^\circledcirc}$ -supermodule on $\mathcal{D}^{\otimes 2}$ via the isomorphism $\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D} \xrightarrow{\sim} \mathcal{D}^{\otimes 2}$ obtained in Proposition 3.5.1. Let us take an algebraically closed field k and a morphism $v^\circledcirc : \text{Spec}(k) \rightarrow S^\circledcirc$, where we equip $\text{Spec}(k)$ with a log structure pulled-back from $S^{\circledcirc \text{log}}$ via v^\circledcirc . Write $\text{Spec}(k)^{\text{log}}$ for the resulting log scheme and $v^{\circledcirc \text{log}} : \text{Spec}(k)^{\text{log}} \rightarrow S^{\circledcirc \text{log}}$ the morphism extending v^\circledcirc . Also, write $(X_v^{\circledcirc \text{log}}/k^{\text{log}}, \{[\sigma_{i,v}^\circledcirc]\}_{i=1}^r, \mathcal{D}_v)$ for the stable log twisted SUSY₁ curve over $\text{Spec}(k)^{\text{log}}$ defined as the base-change of $\S_1 \mathfrak{X}^{\circledcirc \star}$ via $v^{\circledcirc \text{log}}$. It follows from Theorem 3.1.2 that $H^2((X_v)_b, \mathcal{T}_{X_v^{\circledcirc \star \text{-log}}/k^{\text{log}}}^{\mathcal{D}_v}) = 0$. By replacing v^\circledcirc with the various points of S^\circledcirc , one verifies from [10], Ch. III, Theorem 12.11, (a) (or [16], § 5, Remark 3.21, (c)) that $\mathbb{R}^2 f_{b*}(\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D}) = 0$. (Here, note that although the result of *loc. cit.* deals only with the case of schemes, one may prove, by a similar argument, its analogous assertion for the case of superstacks. Thus, in the proof of this proposition, we apply the result of *loc. cit.* as the result corresponding to the case of superstacks.) Hence, the natural morphism

$$(147) \quad v^{\circledcirc \star}(\mathbb{R}^1 f_{b*}(\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D})) \rightarrow H^1((X_v)_b, \mathcal{T}_{X_v^{\circledcirc \star \text{-log}}/k^{\text{log}}}^{\mathcal{D}_v})$$

is surjective for all v^\circledcirc (cf. [10], Ch. III, Theorem 12.11, (b)) and hence, is an isomorphism. Moreover, the last reference or [16], § 5, Remark 3.21, (c) (applied to the case $i = 1$) shows that $\mathbb{R}^1 f_{b*}(\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D})$ is locally free. The rank of this $\mathcal{O}_{S^\circledcirc}$ -supermodule may be calculated by the dimension of $H^1((X_v)_b, \mathcal{T}_{X_v^{\circledcirc \star \text{-log}}/k^{\text{log}}}^{\mathcal{D}_v})$. But, it follows from (144) and Theorem 3.1.2 that

$$(148) \quad \dim_k(H^1((X_v)_b, (\mathcal{T}_{X_v^{\circledcirc \star \text{-log}}/k^{\text{log}}}^{\mathcal{D}_v})_b)) = 3g - 3 + r$$

and

$$(149) \quad \dim_k(H^1((X_v)_b, (\mathcal{T}_{X_v^{\circledcirc \star \text{-log}}/k^{\text{log}}}^{\mathcal{D}_v})_f)) = 2g - 2 + \frac{r}{2},$$

as desired. The commutativity (with respect to base-change over superschemes over S^\circledcirc) of the formulation of $\mathbb{R}^1 f_*^\circledcirc(\mathcal{T}_{X^\circledcirc \star \text{-log}/S^{\circledcirc \text{log}}}^\mathcal{D})$ follows from the above discussion and the discussion in *loc. cit.*. This completes the proof of the former assertion.

The latter assertion follows from [16], § 5, Remark 3.21, (c). \square

4.4. ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ is isomorphic to $({}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$.

One verifies that the morphisms (135) and (141) obtained previously are the inverse morphisms of each other. Thus, we have the following Proposition 4.4.1. In particular, for each r -pointed λ -stable twisted curve $\underline{\mathfrak{X}}^\star := (f : \underline{X} \rightarrow \underline{S}, \{[\underline{\sigma}_i]\}_{i=1}^r)$ of genus g over a scheme \underline{S} , there exists canonically an equivalence of categories between $\mathfrak{Spin}_{\underline{\mathfrak{X}}^\star}$ and the groupoid of stable log twisted SUSY₁ curves ${}^{\S_1}\mathfrak{X}^{\otimes \star}$ over S^{\log} having $\underline{\mathfrak{X}}^\star$ as the underlying pointed twisted curve.

Proposition 4.4.1.

There exists a canonical isomorphism of fibered categories

$$(150) \quad ({}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log} \xrightarrow{\sim} {}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$$

over ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda}^{\log}$. In particular, $({}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$ may be represented by a proper smooth Deligne-Mumford stack over S_0 of relative dimension $3g - 3 + r$.

5. DEFORMATIONS OF STABLE LOG TWISTED SUSY₁ CURVES

In this final section, we prove the main assertion, i.e., Theorem A. As discussed in § 5.4, the main step of the proof is to construct a canonical fermionic deformation of ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$ ($\cong ({}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$ by Proposition 4.4.1) in a way that a universal stable log twisted SUSY₁ curve exists (uniquely). To this end, we develop (in §§ 5.1-5.2) log smooth deformation theory concerning log twisted SUSY₁ curves. By applying the results obtained in these discussions, one may construct (cf. Corollary 5.3.1) a universal family of stable log twisted SUSY₁ curves over a fermionic deformation of a representation (in the sense of Remark 1.8.2 (i)) of ${}^{\text{tw}}\overline{\mathfrak{M}}_{g,r,\lambda,\text{spin}}^{\log}$. It gives rise to a representation (by a groupoid in $\mathfrak{Sch}_{/S_0}^{\otimes}$) of ${}^{\S_1}\overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ itself (equipped with a natural log structure). This implies immediately the proof of Theorem A, as desired (cf. § 5.4 for the detailed discussion).

5.1. Deformation spaces of stable log twisted SUSY₁ curves.

Let $\tilde{S}^{\otimes \log}$ be an fs log scheme and $S^{\otimes \log}$ be a strict closed subsuperscheme of $\tilde{S}^{\otimes \log}$ determined by a nilpotent superideal $\mathcal{I} \subseteq \mathcal{O}_{\tilde{S}^{\otimes}}$ contained in $\mathcal{N}_{\tilde{S}^{\otimes}}$. Let ${}^{\S_1}\mathfrak{X}^{\otimes \star} := (f^{\otimes \log} : X^{\otimes \log} \rightarrow S^{\otimes \log}, \{[\sigma_i^{\otimes}]\}_{i=1}^r, \mathcal{D})$ be a stable log twisted SUSY₁ curve of type (g, r, λ) over $S^{\otimes \log}$. Write ${}^{\S_1}\underline{\mathfrak{X}}^{\otimes \star} := (f^{\otimes \log} : \underline{X}^{\otimes \log} \rightarrow S_t^{\log}, \{[\underline{\sigma}_i]\}_{i=1}^r, \underline{\mathcal{D}})$ for the base-change of ${}^{\S_1}\mathfrak{X}^{\otimes \star}$ via the strict closed immersion

$\tau_S^{\otimes \log} : S_t^{\log} \rightarrow S^{\otimes \log}$ extending τ_S^{\otimes} . Also, write

$$(151) \quad \text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star})$$

for the set of superconformal isomorphism classes of stable log twisted SUSY₁ curves of type (g, r, λ) over $\tilde{S}^{\otimes \log}$ extending ${}^{\S_1}\mathfrak{X}^{\otimes \star}$.

Proposition 5.1.1 (cf. [15], Lemma 2.4).

Suppose that \tilde{S}^{\otimes} is affine and that $\mathcal{N}_{\tilde{S}^{\otimes}}\mathcal{I} = 0$ (which implies that \mathcal{I} is square nilpotent and may be thought of as an \mathcal{O}_{S_t} -module). Then, $\text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star})$ is nonempty and has a canonical structure

$$(152) \quad \begin{aligned} \text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star}) \times H^1(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star} \cdot \log / S_t^{\log}}^{\mathcal{D}} \otimes_{\underline{f}_b^{-1}(\mathcal{O}_{S_t})} \underline{f}_b^{-1}(\mathcal{I}))_b) &\rightarrow \text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star}) \\ (a, \quad b) &\mapsto a \boxplus b \end{aligned}$$

of affine space modeled on $H^1(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star} \cdot \log / S_t^{\log}}^{\mathcal{D}} \otimes_{\underline{f}_b^{-1}(\mathcal{O}_{S_t})} \underline{f}_b^{-1}(\mathcal{I}))_b)$. Also, if ${}^{\S_1}\tilde{\mathfrak{X}}^{\otimes \star}$ is a stable log twisted SUSY₁ curve in $\text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star})$, then there is no nontrivial superconformal automorphism of ${}^{\S_1}\tilde{\mathfrak{X}}^{\otimes \star}$ over $\tilde{S}^{\otimes \log}$ which restricts to the identity morphism of ${}^{\S_1}\mathfrak{X}^{\otimes \star}$.

Proof. First, we shall prove that $\text{Def}_{\tilde{S}^{\otimes \log}}({}^{\S_1}\mathfrak{X}^{\otimes \star})$ is nonempty. Let us take a collection of data

$$(153) \quad \mathbb{U}_I := \{(Y_{\gamma}^{\otimes \log} \xrightarrow{\pi_{\gamma}^{\otimes \log}} X^{\otimes \log}, U_{\gamma}^{\log}, \eta_{\gamma}^{\otimes \log}, z_{\gamma})\}_{\gamma \in I}$$

(indexed by a set I) obtained by applying Corollary 3.4.4 to our ${}^{\S_1}\mathfrak{X}^{\otimes \star}$. For each $i \in \{1, \dots, r\}$, we shall write I_i for the subset of I consisting of elements γ satisfying that $Y_{\gamma}^{\otimes} \times_{X^{\otimes}, \sigma_i^{\otimes}} \mathbb{A}_{S^{\otimes}}^{0|1} \neq \emptyset$. By Proposition 3.3.2, we may assume, without loss of generality, that for each $\gamma \in I_i$ there exists a pair $(\Sigma^{U_{\gamma}}, \sigma^{U_{\gamma}})$ for which the collection of data

$$(154) \quad \mathbb{U}_{\gamma}^{\star} := (Y_{\gamma}^{\otimes \log}, U_{\gamma}^{\otimes \log}, \eta_{\gamma}^{\otimes \log}, \Sigma^{U_{\gamma}}, \sigma^{U_{\gamma}})$$

satisfies the condition described in Proposition 3.3.2. For each $\gamma \in I$ there exists (since Y_{γ}^{\otimes} is affine) a log supersmooth lifting $\tilde{Y}_{\gamma} := (\tilde{Y}_{\gamma}^{\otimes \log}, \tilde{f}_{\gamma}^{\otimes \log}, \tilde{i}_{Y_{\gamma}}^{\otimes \log})$ of $Y_{\gamma}^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$ (cf. Corollary 2.5.3 (iii)) and a log smooth lifting $\tilde{U}_{\gamma}^{\log}$ of U_{γ}^{\log} over \tilde{S}_b^{\log} together with an isomorphism $\tilde{\eta}_{\gamma}^{\otimes \log} : \tilde{Y}_{\gamma}^{\otimes \log} \xrightarrow{\sim} \tilde{U}_{\gamma}^{\log} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^{\otimes}}^{0|1}$ over $\tilde{S}^{\otimes \log}$ lifting $\eta_{\gamma}^{\otimes \log}$. Also, z_{γ} lifts to an element $\tilde{z}_{\gamma} \in \mathcal{M}_{\tilde{U}_{\gamma}}$. The $\mathcal{O}_{\tilde{U}_{\gamma}^{\log} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^{\otimes}}}$ -subsupermodule $\mathcal{O}_{\tilde{U}_{\gamma}^{\log} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^{\otimes}}} \cdot (\partial_{\psi} + \psi \cdot \partial_{\tilde{z}_{\gamma}}) \subseteq \mathcal{T}_{\tilde{U}_{\gamma}^{\log} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^{\otimes}}} / \tilde{S}^{\otimes \log}$ defines, via $\tilde{\eta}_{\gamma}^{\otimes \log}$, a superconformal structure $\tilde{\mathcal{D}}_{\gamma}$ on $\tilde{Y}_{\gamma}^{\otimes \log} / \tilde{S}^{\otimes \log}$ extending $\mathcal{D}|_{Y_{\gamma}^{\otimes \log}}$. After

possibly replacing \mathbb{U}_I with its refinement in an evident sense (cf. Remark 3.3.4), we may suppose that the following three properties (i)-(iii) are satisfied:

- (i) For each pair (γ_1, γ_2) with $Y_{\gamma_1, \gamma_2}^{\otimes \log} := Y_{\gamma_1}^{\otimes \log} \times_{X^{\otimes \log}} Y_{\gamma_2}^{\otimes \log} \neq \emptyset$ (hence $Y_{\gamma_1, \gamma_2}^{\otimes \log}$ is affine), there exists an isomorphism

$$(155) \quad \nu_{\gamma_1, \gamma_2}^{\otimes \log} : \tilde{Y}_{\gamma_1}^{\otimes \log}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}} \xrightarrow{\sim} \tilde{Y}_{\gamma_2}^{\otimes \log}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}}$$

of log supersmooth liftings which sends $\tilde{\mathcal{D}}_{\gamma_1}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}}$ to $\tilde{\mathcal{D}}_{\gamma_2}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}}$ (cf. Corollary 2.5.3 (ii) and Proposition 3.4.3).

- (ii) If $\gamma \in I_i$ ($i = 1, \dots, r$), then there exists a pair $(\Sigma_i^{\tilde{U}_\gamma}, \sigma_i^{\tilde{U}_\gamma})$ consisting of a scheme $\Sigma_i^{\tilde{U}_\gamma}$ over \tilde{S}_b with $\Sigma_i^{\tilde{U}_\gamma} \times_{\tilde{S}_b} S_b \cong \Sigma_i^{U_\gamma}$ and a closed immersion $\sigma_i^{\tilde{U}_\gamma} : \Sigma_i^{\tilde{U}_\gamma} \rightarrow \tilde{U}_\gamma$ over \tilde{S}_b extending $\sigma_i^{U_\gamma}$;
 (iii) For each pair $(\gamma_1, \gamma_2) \in I_i \times I_i$ ($i = 1, \dots, r$) with $Y_{\gamma_1, \gamma_2}^{\otimes \log} \neq \emptyset$, then the restrictions to $Y_{\gamma_1, \gamma_2}^{\otimes \log}$ of two composites

$$(156) \quad \Sigma_i^{\tilde{U}_\gamma} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^\otimes}^{0|1} \xrightarrow{\sigma_i^{\tilde{U}_\gamma}} \tilde{U}_\gamma \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^\otimes}^{0|1} \xrightarrow{(\tilde{\eta}_\gamma^\otimes)^{-1}} \tilde{Y}_\gamma^\otimes$$

($l = 1, 2$) are compatible (in an evident sense) via $\nu_{\gamma_1, \gamma_2}^{\otimes \log}$.

If a triple $(\gamma_1, \gamma_2, \gamma_3) \in I^{\times 3}$ satisfies that

$$(157) \quad Y_{\gamma_1, \gamma_2, \gamma_3}^{\otimes \log} := Y_{\gamma_1}^{\otimes \log} \times_{X^{\otimes \log}} Y_{\gamma_2}^{\otimes \log} \times_{X^{\otimes \log}} Y_{\gamma_3}^{\otimes \log} \neq \emptyset,$$

then there exists uniquely an element

$$(158) \quad \nu_{\gamma_1, \gamma_2, \gamma_3}^\dagger \in \Gamma((Y_{\gamma_1, \gamma_2, \gamma_3})_b, (\mathcal{T}_{X^{\otimes \star \log}/S^{\otimes \log}}^\mathcal{D} \otimes_{f_b^{-1}(\mathcal{O}_{S^\otimes})} f_b^{-1}(\mathcal{I}))_b) \\ (= \Gamma((Y_{\gamma_1, \gamma_2, \gamma_3})_t, (\mathcal{T}_{\underline{X}^{\otimes \star \log}/S_t^{\otimes \log}}^\mathcal{D} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b))$$

such that

$$(159) \quad \nu_{\gamma_1, \gamma_2}^{\otimes \log} \circ \nu_{\gamma_2, \gamma_3}^{\otimes \log} \circ \nu_{\gamma_3, \gamma_1}^{\otimes \log} = \text{id}_{Y_{\gamma_1, \gamma_2, \gamma_3}^{\otimes \log}} \boxplus^\dagger \nu_{\gamma_1, \gamma_2, \gamma_3}^\dagger.$$

The collection of elements $\{\nu_{\gamma_1, \gamma_2, \gamma_3}^\dagger\}_{\gamma_1, \gamma_2, \gamma_3}$ determines an element

$$(160) \quad \nu^\dagger \in H^2(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log}/S_t^{\otimes \log}}^\mathcal{D} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b).$$

But, since S^\otimes is affine and $\dim(\underline{X}_b/S_t) = 1$, we have

$$(161) \quad H^2(\underline{X}_b, \mathcal{T}_{\underline{X}^{\otimes \star \log}/S_t^{\otimes \log}}^\mathcal{D} \otimes_{\underline{f}_b^{-1}(\mathcal{O}_{S^\otimes})} \underline{f}_b^{-1}(\mathcal{I})) = 0$$

(in particular, $\nu^\dagger = 0$). Thus, after possibly replacing \mathbb{U}_I with its refinement and replacing each $\nu_{\gamma_1, \gamma_2}^{\otimes \log}$ with a suitable isomorphism $\tilde{Y}_{\gamma_1}^{\otimes \log}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}} \xrightarrow{\sim} \tilde{Y}_{\gamma_2}^{\otimes \log}|_{Y_{\gamma_1, \gamma_2}^{\otimes \log}}$, the log superschemes $\{\tilde{Y}_\gamma^{\otimes \log}\}_{\gamma \in I}$ may be glued together to a log supersmooth superstack $\tilde{X}^{\otimes \log}$ over $\tilde{S}^{\otimes \log}$. For each $i \in \{1, \dots, r\}$, the morphisms $\{(\sigma_i^{\tilde{U}_\gamma} \times \text{id}) \circ (\tilde{\eta}_\gamma^\otimes)^{-1} : \Sigma_i^{\tilde{U}_\gamma} \times_{\tilde{S}_b} \mathbb{A}_{\tilde{S}^\otimes}^{0|1} \rightarrow \tilde{Y}_\gamma^\otimes\}_{\gamma \in I}$ may be glued together to a closed

immersion $\tilde{\sigma}_i^{\otimes} : \mathbb{A}_{\tilde{S}^{\otimes}}^{0|1} \rightarrow \tilde{X}^{\otimes}$ over \tilde{S}^{\otimes} extending σ_i^{\otimes} , for which the collection of data $(\tilde{X}^{\otimes \log} / \tilde{S}^{\otimes \log}, \{\tilde{\sigma}_i^{\otimes}\}_{i=1}^r)$ forms an r -pointed log twisted $(1|1)$ -curve of genus g over $\tilde{S}^{\otimes \log}$. Moreover, $\{\tilde{\mathcal{D}}_\gamma\}_{\gamma \in I}$ may be glued together to a superconformal structure $\tilde{\mathcal{D}}$ on $\tilde{X}^{\otimes \star \log} / \tilde{S}^{\otimes \log}$ extending \mathcal{D} . The collection of data

$$(162) \quad {}_{\S 1} \tilde{\mathfrak{X}}^{\otimes \star} := (\tilde{X}^{\otimes \log} / \tilde{S}^{\otimes \log}, \{[\tilde{\sigma}_i^{\otimes}]\}_{i=1}^r, \tilde{\mathcal{D}})$$

forms a stable log twisted SUSY₁ curve of type (g, r, λ) over $\tilde{S}^{\otimes \log}$ which restricts to ${}_{\S 1} \mathfrak{X}^{\otimes \star}$. Consequently, $\text{Def}_{\tilde{S}^{\otimes \log}}({}_{\S 1} \mathfrak{X}^{\otimes \log})$ is nonempty.

Also, by considering the above discussion and a usual discussion in deformation theory, $\text{Def}_{\tilde{S}^{\otimes \log}}({}_{\S 1} \mathfrak{X}^{\otimes \log})$ admits a structure of affine space modeled on $H^1(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}^{\underline{\mathcal{D}}} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b)$ as described in (152).

Finally, the above argument implies the remaining portion of the proposition. Indeed, the group of superconformal automorphisms of an arbitrary ${}_{\S 1} \tilde{\mathfrak{X}}^{\otimes \log} \in \text{Def}_{\tilde{S}^{\otimes \log}}({}_{\S 1} \mathfrak{X}^{\otimes \log})$ is canonically isomorphic to $H^0(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}^{\underline{\mathcal{D}}} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b)$. If $(\mathcal{L}^{\star}, \eta^{\star})$ denotes the spin structure on $(\underline{X}_t / S_t, \{[\underline{\sigma}_i]\}_{i=1}^r)$ corresponding to $\underline{\mathcal{D}}$ (cf. Proposition 4.4.1), then we obtain a sequence of isomorphisms

$$(163) \quad \begin{aligned} & (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}^{\underline{\mathcal{D}}} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b \\ & \xrightarrow{\sim} (\underline{\mathcal{D}}^{\otimes 2} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b \\ & \xrightarrow{\sim} ((\mathcal{O}_{\underline{X}^{\otimes}} \otimes \mathcal{L}^{\star \vee})^{\otimes 2} \otimes_{(\underline{f}_t)^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b \\ & \xrightarrow{\sim} (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_b)) \oplus (\mathcal{L}^{\star \vee} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_f)), \end{aligned}$$

where the first isomorphism follows from Proposition 3.5.1 and the second isomorphism follows from (139). Here, note that the natural morphisms

$$(164) \quad H^0(\underline{X}_b, \mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}} \otimes_{\Gamma(S_t, \mathcal{O}_{S_t})} \Gamma(S_t, \mathcal{I}_b)) \rightarrow H^0(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_b)))$$

and

$$(165) \quad H^0(\underline{X}_b, \mathcal{L}^{\star \vee}) \otimes_{\Gamma(S_t, \mathcal{O}_{S_t})} \Gamma(S_t, \mathcal{I}_f) \rightarrow H^0(\underline{X}_b, (\mathcal{L}^{\star \vee} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_f)))$$

are surjective. On the other hand, the fact that $\deg(\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}) < 0$ and $\deg(\mathcal{L}^{\star \vee}) < 0$ implies the equalities $H^0(\underline{X}_b, \mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}) = H^0(\underline{X}_b, \mathcal{L}^{\star \vee}) = 0$. Hence, we have

$$(166) \quad \begin{aligned} & H^0(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}}^{\underline{\mathcal{D}}} \otimes_{\underline{f}_t^{-1}(\mathcal{O}_{S_t})} \underline{f}_t^{-1}(\mathcal{I}))_b) \\ & = H^0(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes \star \log} / S_t^{\log}} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_b))) \oplus H^0(\underline{X}_b, (\mathcal{L}^{\star \vee} \otimes_{\underline{f}_t^*(\mathcal{O}_{S_t})} \underline{f}_t^*(\mathcal{I}_f))) \\ & = 0. \end{aligned}$$

This implies that there is no nontrivial superconformal automorphism of any ${}_{\S_1}\tilde{\mathfrak{X}}^{\otimes\star} \in \text{Def}_{\tilde{S}^{\otimes\log}}({}_{\S_1}\mathfrak{X}^{\otimes\log})$. This completes the proof of Proposition 5.1.1. \square

5.2. Deformations of morphisms.

We keep the notation in the previous subsection. Moreover, let $T^{\otimes\log}$ be an fs log superscheme which is log supersmooth over S_0 (of some relative superdimension) and ${}_{\S_1}\mathfrak{Y}^{\otimes\star} := (f'^{\otimes\log} : Y^{\otimes\log} \rightarrow T^{\otimes\log}, \{[\sigma'_i]\}_{i=1}^r, \mathcal{D}')$ a stable log twisted SUSY₁ curve of type (g, r, λ) over $T^{\otimes\log}$ such that $\mathcal{KS}({}_{\S_1}\mathfrak{Y}^{\otimes\star})$ is an isomorphism. Suppose that we are given a morphism $s^{\otimes\log} : S^{\otimes\log} \rightarrow T^{\otimes\log}$ of log superschemes via which the base-change of ${}_{\S_1}\mathfrak{Y}^{\otimes\star}$ is isomorphic to ${}_{\S_1}\mathfrak{X}^{\otimes\star}$ over $S^{\otimes\log}$.

The following proposition is immediately verified from the various definitions involved, including the affine structures described in Proposition 2.4.3 (ii) and Proposition 5.1.1.

Proposition 5.2.1 (cf. [15], Lemma 2.5).

Suppose that \tilde{S}^{\otimes} is affine and that $\mathcal{N}_{\tilde{S}^{\otimes}}\mathcal{I} = 0$. Denote by $\mathcal{KS}({}_{\S_1}\mathfrak{Y}^{\otimes\star}; \mathcal{I})$ the composite isomorphism

$$\begin{aligned} (167) \quad \Gamma(S_b, (s^{\otimes*}(\mathcal{T}_{T^{\otimes\log}/S_0}) \otimes \mathcal{I})_b) &\xrightarrow{\sim} \Gamma(S_b, (s^{\otimes*}(\mathbb{R}^1 f'_{b*}(\mathcal{T}_{Y^{\otimes\log}/T^{\otimes\log}}^{\mathcal{D}})) \otimes \mathcal{I})_b) \\ &\xrightarrow{\sim} \Gamma(S_b, (\mathbb{R}^1 f_{b*}(\mathcal{T}_{\underline{X}^{\otimes\star-\log}/S_t^{\otimes\log}}^{\mathcal{D}}) \otimes \mathcal{I})_b) \\ &\xrightarrow{\sim} H^1(\underline{X}_b, (\mathcal{T}_{\underline{X}^{\otimes\star-\log}/S_t^{\otimes\log}}^{\mathcal{D}} \otimes \underline{f}_b^*(\mathcal{O}_{S_t}) \underline{f}_b^*(\mathcal{I}))_b), \end{aligned}$$

where the first isomorphism arises from $\mathcal{KS}({}_{\S_1}\mathfrak{Y}^{\otimes\star})$ and both the second and third isomorphisms arise from Proposition 4.3.2. Consider the map of sets

$$(168) \quad s^{\otimes*} : \Gamma(\tilde{S}_b, \text{Def}_{\tilde{S}^{\otimes\log}}(s^{\otimes\log})) \rightarrow \text{Def}_{\tilde{S}^{\otimes\log}}({}_{\S_1}\mathfrak{X}^{\otimes\star})$$

given by pulling-back ${}_{\S_1}\mathfrak{Y}^{\otimes\star}$. Then, this map satisfies the equality

$$(169) \quad s^{\otimes*}(\tilde{s}^{\otimes\log} \boxplus^{\dagger} \zeta) = s^{\otimes*}(\tilde{s}^{\otimes\log}) \boxplus^{\dagger} \mathcal{KS}({}_{\S_1}\mathfrak{Y}^{\otimes\star}; \mathcal{I})(\zeta).$$

for any $\tilde{s}^{\otimes\log} \in \Gamma(\tilde{S}_b, \text{Def}_{\tilde{S}^{\otimes\log}}(s^{\otimes\log}))$ and $\zeta \in \Gamma(S_b, (s^{\otimes*}(\mathcal{T}_{T^{\otimes\log}/S_0}) \otimes \mathcal{I})_b)$. In particular, (since $\mathcal{KS}({}_{\S_1}\mathfrak{Y}^{\otimes\star}; \mathcal{I})$ is an isomorphism) $s^{\otimes*}$ is bijective, and hence, $\Gamma(\tilde{S}_b, \text{Def}_{\tilde{S}^{\otimes\log}}(s^{\otimes\log}))$ is nonempty.

Corollary 5.2.2 (cf. [15], Theorem 2.7).

Suppose that we are given a stable log twisted SUSY₁ curve ${}_{\S_1}\tilde{\mathfrak{X}}^{\otimes\star}$ over $\tilde{S}^{\otimes\log}$ extending ${}_{\S_1}\mathfrak{X}^{\otimes\star}$. Then, there exists uniquely an extension $\tilde{s}^{\otimes\log} : \tilde{S}^{\otimes\log} \rightarrow T^{\otimes\log}$ of $s^{\otimes\log}$ via which the base-change of ${}_{\S_1}\mathfrak{Y}^{\otimes\log}$ is isomorphic to ${}_{\S_1}\tilde{\mathfrak{X}}^{\otimes\star}$.

Proof. The assertion may be directly proved by applying inductively Proposition 5.2.1 to the case where the pair $(S^{\otimes \log}, \tilde{S}^{\otimes \log})$ is taken to be $(\tilde{S}_n^{\otimes \log}, \tilde{S}_{n+1}^{\otimes \log})$ (where, for each $n \geq 0$, we shall denote by $\tilde{S}_n^{\otimes \log}$ the strict closed subsuperscheme of $\tilde{S}^{\otimes \log}$ determined by $\mathcal{N}_{\tilde{S}^{\otimes \log}}^{n+1}$). \square

5.3. Canonical liftings over complete versal families.

Proposition 5.3.1 (cf. [15], Theorem 2.8).

Let \underline{T}^{\log} be an affine log smooth scheme in $\mathfrak{Sch}_{/S_0}^{\log}$, and let ${}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \log} := (\underline{f}^{\otimes \log} : \underline{Y}^{\otimes \log} \rightarrow \underline{T}^{\log}, \{[\underline{\sigma}_i]\}_{i=1}^r, \underline{\mathcal{D}})$ be a stable log twisted SUSY₁ curve of type (g, r, λ) over \underline{T}^{\log} such that the classical Kodaira-Spencer map

$$(170) \quad \mathcal{KS}(\underline{Y}_b^{\star-\log}/\underline{T}^{\log}) : \mathcal{T}_{\underline{T}^{\log}/S_0} \rightarrow \mathbb{R}^1 \underline{f}_{b*}(\mathcal{T}_{\underline{Y}_b^{\star-\log}/\underline{T}^{\log}})$$

of $\underline{Y}_b^{\star-\log} \rightarrow \underline{T}^{\log}$ (cf. (118)) is an isomorphism. Let us write

$$(171) \quad \langle \underline{T} \rangle^{\otimes \log} := \langle \underline{T}, \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \log}/\underline{T}^{\log}})^\vee) \rangle^{\otimes \log}.$$

Then, there exists a stable log twisted SUSY₁ curve ${}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star}$ of type (g, r, λ) over $\langle \underline{T} \rangle^{\otimes \log}$ which restricts to ${}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \star}$ via $\tau_{\langle \underline{T} \rangle}^{\otimes \log} : \underline{T}^{\log} \rightarrow \langle \underline{T} \rangle^{\otimes \log}$ and whose Kodaira-Spencer map $\mathcal{KS}({}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star})$ is an isomorphism. Moreover, such a stable log twisted SUSY₁ curve is unique in the following sense: if ${}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star}$ and ${}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star}$ are stable log twisted SUSY₁ curves of type (g, r, λ) over $\langle \underline{T} \rangle^{\otimes \log}$ which restrict to ${}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \star}$ and whose Kodaira-Spencer maps are isomorphism, then there exists uniquely a superconformal isomorphism ${}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star} \xrightarrow{\sim} {}^{\S_1}\mathfrak{Y}_{\dagger}^{\otimes \star}$ over $\langle \underline{T} \rangle^{\otimes \log}$ which restricts to the identity morphism of ${}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \star}$.

Proof. The uniqueness portion follows from the uniqueness assertion of Corollary 5.2.2. We shall prove the existence portion. For each nonnegative integer n , we shall write $\langle \underline{T} \rangle_n^{\otimes \log}$ for the strict closed subsuperscheme of $\langle \underline{T} \rangle^{\otimes \log}$ corresponding to the ideal $\mathcal{N}_{\langle \underline{T} \rangle^{\otimes \log}}^{n+1}$. Since $\langle \underline{T} \rangle_1^{\otimes \log}$ is simply $(\underline{T}, \mathcal{O}_{\underline{T}} \oplus \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \log}/\underline{T}^{\log}})^\vee))$, we obtain the trivial deformation ${}^{\S_1}\mathfrak{Y}_{1, \text{triv}}^{\otimes \star}$ of ${}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \star}$ over $\langle \underline{T} \rangle_1^{\otimes \log}$ by pulling-back via the projection $\langle \underline{T} \rangle_1^{\otimes \log} \rightarrow \underline{T}^{\log}$. By applying Proposition 5.2.1 and considering the point of the affine space $\text{Def}_{\langle \underline{T} \rangle_1^{\otimes \log}}({}^{\S_1}\underline{\mathfrak{Y}}^{\otimes \star})$ representing ${}^{\S_1}\mathfrak{Y}_{1, \text{triv}}^{\otimes \star}$ as

its origin, we have a canonical composite bijection

$$\begin{aligned}
(172) \quad & \text{Def}_{\langle \underline{T} \rangle_1^{\otimes \log}}({}^{\S 1}\mathfrak{Y}_1^{\otimes \star}) \\
& \xrightarrow{\sim} H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}} \otimes \underline{f}_b^*(\mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)^\vee))_b) \\
& \xrightarrow{\sim} H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f \otimes \underline{f}_b^*(\mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)^\vee)) \\
& \xrightarrow{\sim} H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f) \otimes_{H^0(\underline{T}, \mathcal{O}_{\underline{T}})} H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)^\vee \\
& \xrightarrow{\sim} \text{End}_{H^0(\underline{T}, \mathcal{O}_{\underline{T}})}(H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)),
\end{aligned}$$

where the third bijection follows from Proposition 4.3.2. If we write ${}^{\S 1}\mathfrak{Y}_1^{\otimes \star}$ for the stable log twisted SUSY₁ curve corresponding to

$$(173) \quad \text{id}_{H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)} \in \text{End}_{H^0(\underline{T}, \mathcal{O}_{\underline{T}})}(H^1(\underline{Y}_b, (\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f))$$

via (172), then its Kodaira-Spencer map turns out to be an isomorphism. By Proposition 5.1.1, ${}^{\S 1}\mathfrak{Y}_1^{\otimes \star}$ may be deformed to a stable log twisted SUSY₁ curve

$$(174) \quad {}^{\S 1}\mathfrak{Y}_\dagger^{\otimes \star} := (f_\dagger^{\otimes \log} : Y_\dagger^{\otimes \log} \rightarrow \langle \underline{T} \rangle^{\otimes \log}, \{[\sigma_{\dagger, i}^{\otimes}]\}_{i=1}^r, \mathcal{D}_\dagger)$$

of type (g, r, λ) over $\langle \underline{T} \rangle^{\otimes \log}$.

We shall prove that the Kodaira-Spencer map $\mathcal{KS}({}^{\S 1}\mathfrak{Y}_\dagger^{\otimes \star})$ is an isomorphism. To this end, it suffices to prove that its restriction along the reduced space \underline{T} is an isomorphism. (Indeed, by Proposition 4.3.2, $\mathbb{R}^1 f_{\dagger*}^{\otimes}(\mathcal{T}_{Y_\dagger^{\otimes \star - \log} / \langle \underline{T} \rangle^{\otimes \log}}^{\mathcal{D}_\dagger})$ and $\mathcal{T}_{\langle \underline{T} \rangle^{\otimes \log} / S_0}$ are locally free of the same rank). By the definition of $\langle \underline{T} \rangle^{\otimes \log}$, the pull-back of $\mathcal{T}_{\langle \underline{T} \rangle^{\otimes \log} / S_0}$ via $\tau_{\langle \underline{T} \rangle}^{\otimes} : \underline{T} \rightarrow \langle \underline{T} \rangle$ admits a canonical isomorphism

$$(175) \quad \tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathcal{T}_{\langle \underline{T} \rangle^{\otimes \log} / S_0}) \xrightarrow{\sim} \mathcal{T}_{\underline{T}^{\log} / S_0} \oplus \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f).$$

On the other hand, the pull-back of $\mathbb{R}^1 f_{\dagger*}^{\otimes}(\mathcal{T}_{Y_\dagger^{\otimes \star - \log} / \langle \underline{T} \rangle^{\otimes \log}}^{\mathcal{D}_\dagger})$ admits a canonical composite isomorphism

$$\begin{aligned}
(176) \quad & \tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathbb{R}^1 f_{\dagger*}^{\otimes}(\mathcal{T}_{Y_\dagger^{\otimes \star - \log} / \langle \underline{T} \rangle^{\otimes \log}}^{\mathcal{D}_\dagger})) \\
& \xrightarrow{\sim} \mathbb{R}^1 \underline{f}_*^{\otimes}(\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}}) \\
& \xrightarrow{\sim} \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_b) \oplus \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f) \\
& \xrightarrow{\sim} \mathbb{R}^1 \underline{f}_{b*}(\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}) \oplus \mathbb{R}^1 \underline{f}_*^{\otimes}((\mathcal{T}_{\underline{Y}^{\otimes \star - \log} / \underline{T}^{\log}}^{\mathcal{D}})_f)
\end{aligned}$$

where the first isomorphism follows from Proposition 4.3.2 and the third isomorphism follows from (143). One may verify immediately from the various definitions involved that the pull-back $\tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathcal{KS}({}^{\S 1}\mathfrak{Y}_\dagger^{\otimes \star}))$ of $\mathcal{KS}({}^{\S 1}\mathfrak{Y}_\dagger^{\otimes \star})$ makes

the square diagram

(177)

$$\begin{array}{ccc}
 \tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathcal{T}_{\langle \underline{T} \rangle^{\otimes \log}/S_0}) & \xrightarrow[\sim]{(175)} & \mathcal{T}_{\underline{T}^{\log}/S_0} \oplus \mathbb{R}^1 f_{b*}((\mathcal{T}_{\underline{Y}^{\otimes*-\log}/\underline{T}^{\otimes \log}})^{\mathcal{D}})_f) \\
 \tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathcal{KS}(\mathfrak{S}_1 \mathfrak{Y}_{\dagger}^{\otimes*})) \downarrow & & \downarrow \mathcal{KS}(\underline{Y}_b^{\star-\log}/\underline{T}^{\log}) \oplus \text{id} \\
 \tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathbb{R}^1 f_{\dagger*}(\mathcal{T}_{\underline{Y}_{\dagger}^{\otimes*-\log}/\langle \underline{T} \rangle^{\otimes \log}})^{\mathcal{D}_{\dagger}})) & \xrightarrow[\sim]{(176)} & \mathbb{R}^1 \underline{f}_{b*}(\mathcal{T}_{\underline{Y}_b^{\star-\log}/\underline{T}^{\log}}) \oplus \mathbb{R}^1 \underline{f}_{b*}((\mathcal{T}_{\underline{Y}^{\otimes*-\log}/\underline{T}^{\otimes \log}})^{\mathcal{D}})_f)
 \end{array}$$

commute. Hence, since we have assumed that $\mathcal{KS}(\underline{Y}^{\star-\log}/\underline{T}^{\log})$ is an isomorphism, $\tau_{\langle \underline{T} \rangle}^{\otimes*}(\mathcal{KS}(\mathfrak{S}_1 \mathfrak{Y}_{\dagger}^{\otimes*}))$ is an isomorphism, as desired. This completes the proof of Proposition 5.3.1. \square

5.4. The proof of Theorem A.

In this final section, we shall prove Theorem A, the main result of the present paper. Since $(\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t$ is a smooth Deligne-Mumford stack over S_0 (cf. Proposition 4.4.1), there exists an isomorphism $[\underline{R} \rightrightarrows \underline{U}] \xrightarrow{\sim} (\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t$ over S_0 for some groupoid $\underline{R} \rightrightarrows \underline{U} := (\underline{U}, \underline{R}, \underline{s}, \underline{t}, \underline{c})$ in $\mathfrak{Sch}/_{S_0}$ such that both \underline{U} and \underline{R} are smooth affine schemes over S_0 of relative dimension $3g - 3 + r$, and both \underline{s} and \underline{t} are étale. Denote by $\pi_{\underline{U}} : \underline{U} \rightarrow (\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t$ the natural projection (hence $\pi_{\underline{R}} := \pi_{\underline{U}} \circ \underline{s} = \pi_{\underline{U}} \circ \underline{t}$). Write \underline{U}^{\log} (resp., \underline{R}^{\log}) for the log scheme defined to be \underline{U} (resp., \underline{R}) equipped with the log structure pulled-back from $(\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$. In particular, $\pi_{\underline{U}}$ (resp., $\pi_{\underline{R}}$) extends to a morphisms $\pi_{\underline{U}}^{\log} : \underline{U}^{\log} \rightarrow (\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$ (resp., $\pi_{\underline{R}}^{\log} : \underline{R}^{\log} \rightarrow (\mathfrak{S}_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$) of log stacks. Moreover, $\underline{s}, \underline{t} : \underline{R} \rightarrow \underline{U}$ extend to morphisms $\underline{s}^{\log}, \underline{t}^{\log} : \underline{R}^{\log} \rightarrow \underline{U}^{\log}$, and \underline{c} extends to a morphism $\underline{c}^{\log} : \underline{R}^{\log} \times_{\underline{s}^{\log}, \underline{U}^{\log}, \underline{t}^{\log}} \underline{R}^{\log} \rightarrow \underline{R}^{\log}$.

Let us write

$$\begin{aligned}
 (178) \quad \mathfrak{S}_1 \underline{\mathfrak{Y}}^{\otimes*} &:= (\underline{Y}^{\otimes \log}/\underline{U}^{\log}, \{[\sigma_{\underline{Y},i}^{\otimes}]\}_{i=1}^r, \mathcal{D}_{\underline{Y}}) \\
 (\text{resp., } \mathfrak{S}_1 \underline{\mathfrak{X}}^{\otimes*} &:= (\underline{X}^{\otimes \log}/\underline{R}^{\log}, \{[\sigma_{\underline{X},i}^{\otimes}]\}_{i=1}^r, \mathcal{D}_{\underline{X}})
 \end{aligned}$$

for the stable log twisted SUSY₁ curve over \underline{U}^{\log} (resp., \underline{R}^{\log}) classified by $\pi_{\underline{U}}^{\log}$ (resp., $\pi_{\underline{R}}^{\log}$). The base-change of $\mathfrak{S}_1 \underline{\mathfrak{Y}}^{\otimes*}$ via \underline{s}^{\log} and \underline{t}^{\log} respectively are, by definition, isomorphic to $\mathfrak{S}_1 \underline{\mathfrak{X}}^{\otimes*}$. The Kodaira-Spencer morphisms $\mathcal{KS}(\mathfrak{S}_1 \underline{\mathfrak{Y}}^{\otimes*})$, $\mathcal{KS}(\mathfrak{S}_1 \underline{\mathfrak{X}}^{\otimes*})$ are isomorphisms. Here, let us define $\langle \underline{U} \rangle^{\otimes \log}$ (resp., $\langle \underline{R} \rangle^{\otimes \log}$) to be the log superscheme obtained from \underline{U}^{\log} (resp., \underline{R}^{\log}) as in (171), which is split and log supersmooth over S_0 of relative superdimension $3g - 3 + r|2g - 2 + \frac{r}{2}$ (by Proposition 4.3.2). It follows from Proposition 5.3.1 that $\mathfrak{S}_1 \underline{\mathfrak{Y}}^{\otimes*}$ (resp.,

$\S_1 \underline{\mathfrak{X}}^{\otimes \star})$ may be deformed to a stable log twisted SUSY₁ curve

$$(179) \quad \begin{aligned} \S_1 \mathfrak{Y}_{\dagger}^{\otimes \star} &:= (f_{Y,\dagger}^{\otimes \log} : Y_{\dagger}^{\otimes \log} \rightarrow \langle \underline{U} \rangle^{\otimes \log}, \{[\sigma_{Y,\dagger,i}^{\otimes \log}]\}_{i=1}^r, \mathcal{D}_{Y,\dagger}) \\ (\text{resp.}, \S_1 \mathfrak{X}_{\dagger}^{\otimes \star} &:= (f_{X,\dagger}^{\otimes \log} : X_{\dagger}^{\otimes \log} \rightarrow \langle \underline{R} \rangle^{\otimes \log}, \{[\sigma_{X,\dagger,i}^{\otimes \log}]\}_{i=1}^r, \mathcal{D}_{X,\dagger})) \end{aligned}$$

over $\langle \underline{U} \rangle^{\otimes \log}$ (resp., $\langle \underline{R} \rangle^{\otimes \log}$) whose Kodaira-Spencer map is an isomorphism. Hence, by Corollary 5.2.2, there exists morphisms $\langle \underline{s} \rangle^{\otimes \log}, \langle \underline{t} \rangle^{\otimes \log} : \langle \underline{R} \rangle^{\otimes \log} \rightarrow \langle \underline{U} \rangle^{\otimes \log}$ via which the base-changes of $\S_1 \mathfrak{Y}_{\dagger}^{\otimes \star}$ are isomorphic to $\S_1 \mathfrak{X}_{\dagger}^{\otimes \star}$ and which make the square diagrams

$$(180) \quad \begin{array}{ccc} \underline{R}^{\log} & \xrightarrow{\underline{s}^{\log}} & \underline{U}^{\log} \\ \tau_{\langle \underline{R} \rangle}^{\otimes \log} \downarrow & & \downarrow \tau_{\langle \underline{U} \rangle}^{\otimes \log} \\ \langle \underline{R} \rangle^{\otimes \log} & \xrightarrow{\langle \underline{s} \rangle^{\otimes \log}} & \langle \underline{U} \rangle^{\otimes \log} \end{array} \quad \begin{array}{ccc} \underline{R}^{\log} & \xrightarrow{\underline{t}^{\log}} & \underline{U}^{\log} \\ \tau_{\langle \underline{R} \rangle}^{\otimes \log} \downarrow & & \downarrow \tau_{\langle \underline{U} \rangle}^{\otimes \log} \\ \langle \underline{R} \rangle^{\otimes \log} & \xrightarrow{\langle \underline{t} \rangle^{\otimes \log}} & \langle \underline{U} \rangle^{\otimes \log} \end{array}$$

commute. Moreover, we obtain a morphism

$$(181) \quad \langle \underline{c} \rangle^{\otimes \log} : \langle \underline{R} \rangle^{\otimes \log} \times_{\langle \underline{s} \rangle^{\otimes \log}, \langle \underline{U} \rangle^{\otimes \log}, \langle \underline{t} \rangle^{\otimes \log}} \langle \underline{R} \rangle^{\otimes \log} \rightarrow \langle \underline{R} \rangle^{\otimes \log}$$

extending the morphism $\underline{c}^{\otimes \log}$. The uniqueness assertion in Corollary 5.2.2 implies that the collection of data

$$(182) \quad \langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes} := (\langle \underline{U} \rangle^{\otimes}, \langle \underline{R} \rangle^{\otimes}, \langle \underline{s} \rangle^{\otimes}, \langle \underline{t} \rangle^{\otimes}, \langle \underline{c} \rangle^{\otimes})$$

forms a groupoid in $\mathfrak{Sch}_{/S_0}^{\otimes}$.

For $\square = \underline{s}$ or \underline{t} , we shall denote by

$$(183) \quad d\langle \square \rangle^{\otimes} : \mathcal{T}_{\langle \underline{R} \rangle^{\otimes \log}/S_0} \rightarrow \langle \square \rangle^{\otimes *}(\mathcal{T}_{\langle \underline{U} \rangle^{\otimes \log}/S_0})$$

the differential of $\langle \square \rangle^{\otimes}$ relative to S_0 . Then, $d\langle \square \rangle^{\otimes}$ is an isomorphism since the square diagram

$$(184) \quad \begin{array}{ccc} \mathcal{T}_{\langle \underline{R} \rangle^{\otimes \log}/S_0} & \xrightarrow{d\langle \square \rangle^{\otimes}} & \langle \square \rangle^{\otimes *}(\mathcal{T}_{\langle \underline{U} \rangle^{\otimes \log}/S_0}) \\ \downarrow \wr \mathcal{KS}(\S_1 \mathfrak{X}_{\dagger}^{\otimes \star}) & & \downarrow \wr \langle \square \rangle^{\otimes *}(\mathcal{KS}(\S_1 \mathfrak{Y}_{\dagger}^{\otimes \star})) \\ \mathbb{R}^1 f_{X,\dagger}^{\otimes}(\mathcal{T}_{X_{\dagger}^{\otimes \star-\log}/\langle \underline{R} \rangle^{\otimes \log}}^{\mathcal{D}_{X,\dagger}}) & \xrightarrow{\sim} & \langle \square \rangle^{\otimes *}(\mathbb{R}^1 f_{Y,\dagger}^{\otimes}(\mathcal{T}_{Y_{\dagger}^{\otimes \star-\log}/\langle \underline{U} \rangle^{\otimes \log}}^{\mathcal{D}_{Y,\dagger}})) \end{array}$$

is commute and cartesian (where the lower horizontal arrow is isomorphism by Proposition 4.3.2). Hence, for each $n \geq 0$, the morphism $\text{gr}_{\langle \underline{U} \rangle^{\otimes}}^n \rightarrow \text{gr}_{\langle \underline{R} \rangle^{\otimes}}^n$ induced by $d\langle \square \rangle^{\otimes}$ is an isomorphism. It follows immediately that $\langle \underline{s} \rangle_b, \langle \underline{t} \rangle_b : \langle \underline{R} \rangle_b \rightarrow \langle \underline{U} \rangle_b$ are étale (since \underline{s} and \underline{t} are étale) and that two morphisms

$$(185) \quad (\langle \underline{s} \rangle^{\otimes}, \beta_{\langle \underline{R} \rangle_b}^{\otimes}), (\langle \underline{t} \rangle^{\otimes}, \beta_{\langle \underline{R} \rangle_b}^{\otimes}) : \langle \underline{R} \rangle^{\otimes} \rightarrow \langle \underline{U} \rangle^{\otimes} \times_{\langle \underline{U} \rangle_b} \langle \underline{R} \rangle_b$$

are isomorphisms. Thus, both $\langle \underline{s} \rangle^{\otimes}$ and $\langle \underline{t} \rangle^{\otimes}$ are superétale. By Proposition 1.8.3, $[\langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes}]$ forms a supersmooth Deligne-Mumford superstack over

S_0 of relative superdimension $3g - 3 + r|2g - 2 + \frac{r}{2}$; it is superproper over S_0 since $(\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t$ is proper over S_0 in the classical sense.

Moreover, the log structures of the various constituents in $\langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes}$ gives rises to a log structure on the superstack $[\langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes}]$. Let us write $[\langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes}]^{\log}$ for the resulting log superstack and write $\pi_{\langle \underline{U} \rangle}^{\otimes \log} : \langle \underline{U} \rangle^{\otimes \log} \rightarrow \S_1 \overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ for the classifying morphism of $\S \mathfrak{Y}_{\dagger}^{\otimes \star}$. Then, $\pi_{\langle \underline{U} \rangle}^{\otimes \log}$ factors through a morphism

$$(186) \quad \Theta^{\otimes \log} : [\langle \underline{R} \rangle^{\otimes} \rightrightarrows \langle \underline{U} \rangle^{\otimes}]^{\log} \rightarrow \S_1 \overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}.$$

To complete the proof of Theorem A, it suffices to prove that $\Theta^{\otimes \log}$ is an isomorphism.

Consider the surjective portion. Let $S^{\otimes \log}$ be an object in $\mathfrak{Sch}_{/S_0}^{\otimes \log}$ and $s^{\otimes \log} : S^{\otimes \log} \rightarrow \S_1 \overline{\mathfrak{M}}_{g,r,\lambda}^{\otimes \log}$ a morphism of log superstacks, which induces a morphism $s_t^{\log} : S_t^{\log} \rightarrow (\S_1 \overline{\mathfrak{M}}_{g,r,\lambda})_t^{\log}$. There exists a strict étale covering $\pi_{\underline{s}}^{\log} : \underline{S}'^{\log} \rightarrow S_t^{\log}$ of S_t^{\log} and a morphism $\underline{s}'^{\log} : \underline{S}'^{\log} \rightarrow \underline{U}^{\log}$ satisfying that $s_t^{\log} \circ \pi_{\underline{s}}^{\log} \cong \pi_{\underline{U}}^{\log} \circ \underline{s}'^{\log}$. By Proposition 1.2.5, there exists a strict superétale morphism $\pi_s^{\otimes \log} : S'^{\otimes \log} \rightarrow S^{\otimes \log}$ which fits into the following cartesian square diagram

$$(187) \quad \begin{array}{ccc} \underline{S}'^{\log} & \xrightarrow{\pi_{\underline{s}}^{\log}} & S_t^{\log} \\ \downarrow & \square & \downarrow \tau_S^{\otimes \log} \\ S'^{\otimes \log} & \xrightarrow{\pi_s^{\otimes \log}} & S^{\otimes \log}. \end{array}$$

(In particular, the left-hand vertical arrow coincides with $\tau_{S'}^{\otimes \log}$.) By Corollary 5.2.2, the morphism \underline{s}'^{\log} extends to a morphism $s'^{\otimes \log} : S'^{\otimes \log} \rightarrow \langle \underline{U} \rangle^{\otimes \log}$. The uniqueness assertion of Corollary 5.2.2 implies that $\pi_{\langle \underline{U} \rangle}^{\otimes \log} \circ s'^{\otimes \log} \cong s^{\otimes \log} \circ \pi_s^{\otimes \log}$. This shows the subjectivity of $\Theta^{\otimes \log}$.

The injectivity portion follows from an argument technically similar to the above discussion. This completes the proof of Theorem A.

REFERENCES

- [1] D. Abramovich, Q. Chen, D. Gillam, Y. Huang, M. Olsson, M. Satriano, S. Sun. Logarithmic geometry and moduli. *Handbook of Moduli, Vol. I*, Adv. Lect. Math. **24**, Int. Press, Somerville, MA, (2013), pp. 1-61.
- [2] D. Abramovich, T. J. Jarvis. Moduli of twisted spin curves. *Proc. Amer. Math. Soc.* **131**, (2003), pp. 685-699.
- [3] D. Abramovich, A. Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.* **15**, (2002), pp. 27-75.

- [4] C. Carmeli, L. Caston, R. Fiorese, *Mathematical foundations of supersymmetry*. EMS Ser. Lect. Math., European Math. Soc., Zurich, (2011).
- [5] A. Chiodo. Stable twisted curves and their r -spin structures. *Ann. Inst. Fourier (Grenoble)* **58**, (2008), pp. 1635-1689.
- [6] P. Deligne. Letter to Y. Manin. *unpublished*, Princeton, (1987).
- [7] R. Donagi, E. Witten, Supermoduli space is not projected. *arXiv: hep-th/1304.7798v1*, (2013).
- [8] D. Friedan. Notes on string theory and two dimensional conformal field theory. *Unified String Theories* (World Scientific Singapore, 1986), pp. 162-213.
- [9] M. Groechenig, Moduli stacks of maps for supermanifolds. *arXiv: math.AG/1204.4369v2*, (2014).
- [10] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics **52** Springer-Verlag, New York-Heidelberg (1977).
- [11] L. Illusie, An Overview of the Work of K. Fujiwara, K. Kato and C. Nakamura on Logarithmic Etale Cohomology. *Astérisque* **279** (2002), pp. 271-322.
- [12] M. Kapranov, E. Vasserot. Supersymmetry and the formal loop space. *Advance in Math.* **227**, (2011), pp. 1078-1128.
- [13] K. Kato, Logarithmic structures of Fontaine-Illusie, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, p. 191-224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
- [14] G. Laumon, L. Moret-Bailly. *Champs algébriques*. Erg. der Math. und Grenzg., 3. Folge, **39**, Springer, (2000).
- [15] C. LeBrun, M. Rothstein. Moduli of super Riemann surfaces. *Commun. Math. Phys.* **117**, (1988), pp. 159-176.
- [16] Q. Liu, *Algebraic geometry and arithmetic curves*. Oxford Graduate Texts in Mathematics **6**, Oxford University Press, Oxford, (2006).
- [17] Y. I. Manin, *Gauge field theory and complex geometry*. Grundlehren Math. Wiss. **289**, Springer-Verlag, Berlin, (1997).
- [18] A. Ogus, *F-Crystals, Griffiths Transversality, and the Hodge Decomposition*. *Astérisque* **221**, Soc. Math. de France, (1994).
- [19] M. Olsson. On (log) twisted curves. *Compos. Math.* **143**, (2007), pp. 476-494.
- [20] E. Witten, Notes on super Riemann surfaces and their moduli. *arXiv: hep-th/1209.2459v4*, (2013).